THE EXISTENCE OF INVARIANT SUBSPACES[1]

BY LOUIS DE BRANGES AND JAMES ROVNYAK

Communicated by P. R. Halmos, May 4, 1964

We have succeeded in showing that a bounded linear transformation in a Hilbert space always has invariant subspaces. The existence of invariant subspaces was previously known only under complete continuity hypotheses.

THEOREM. *Let T be a bounded linear transformation in a Hilbert space $\mathcal{H}$. Let f be a given element of $\mathcal{H}$ and let h be a given number,*

$$0 \leqq h \leqq \|Tf\|^2.$$

Then there exist projections P_+ and P_- into invariant subspaces for T such that the range of P_+ contains the range of P_-, the orthogonal complement of the range of P_- in the range of P_+ has dimension 0 *or* 1, *and*

$$\|TP_-f\|^2 \leqq h \leqq \|TP_+f\|^2.$$

The proof depends on Livsič's theory of characteristic operator functions [8], [9]. A characteristic operator function is an operator valued analytic function which is associated with the transformation in such a way that invariant subspaces for the transformation correspond to factors of the function. In general, this correspondence is only formal. Two technical problems have to be solved to show the existence of invariant subspaces. The first is to obtain factorization theorems without making a complete continuity hypothesis, and the second is to obtain an isometric inclusion for related Hilbert spaces. The first problem is solved by using a weak compactness theorem for positive definite operator valued functions. The second problem is solved by using functions which are analytic across the boundary of the unit disk. The theory is formulated in terms of formal power series rather than analytic functions.

We choose and fix a *coefficient space.* This is a Hilbert space $\mathcal{C}$ that we treat as a generalization of the complex numbers. The elements of $\mathcal{C}$ will be called *vectors*, and the bounded linear transformations of $\mathcal{C}$ into itself will be called *operators.* The norm of a vector c is written $|c|$. If b is a vector, then $\bar{b}$ is the linear functional on $\mathcal{C}$ such that $\bar{b}a = \langle a, b\rangle$ for every vector a. The adjoint and norm of an operator A are denoted by $\bar{A}$ and $|A|$.

---

[1] Research supported by the Alfred P. Sloan Foundation and the National Science Foundation.