CONVEX PROGRAMMING IN HILBERT SPACE

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This note gives a construction for minimizing certain twice-differentiable functions on a closed convex subset C, of a Hilbert Space, H. The algorithm assumes one can constructively "project" points onto convex sets. A related algorithm may be found in Cheney-Goldstein [1], where a constructive fixed-point theorem is employed to construct points inducing a minimum distance between two convex sets. In certain instances when such projections are not too difficult to construct, say on spheres, linear varieties, and orthants, the method can be effective. For applications to control theory, for example, see Balakrishnan [2], and Goldstein [3].

In what follows P will denote the "projection" operator for the convex set C. This operator, which is well defined and Lipschitzian, assigns to a given point in H its closest point in C (see, e.g., [1]). Take $x \in H$ and $y \in C$. Then $[x-y,\ P(x)-y] \geqq \|P(x)-y\|^2$. In the nontrivial case this inequality is a consequence of the fact that C is supported by a hyperplane through $P(x)$ with normal $x-P(x)$. Let f be a real-valued function on H and x_0 an arbitrary point of C. Let S denote the level set $\{x \in C: f(x) \leqq f(x_0)\}$, and let $\hat{S}$ be any open set containing the convex hull of S. Let $f'(x,\ \cdot) = [\nabla f(x),\ \cdot\,]$ signify the Fréchet derivative of f at x. A point z in C will be called stationary if $P(z-\rho\nabla f(z)) = z$ for all $\rho > 0$; equivalently, when f is convex the linear functional $f'(z,\ \cdot)$ achieves a minimum on C at z.

THEOREM. *Assume f is bounded below. For each $x \in \hat{S}$, h in H and for some $\rho_0 > 0$, assume that $f'(x, h)$ exists in the sense of Fréchet, $f''(x, h, h)$ exists in the sense of Gâteaux, and $|f''(x, h, h)| \leqq \|h\|^2/\rho_0$. Choose σ and ρ_k satisfying $0 < \sigma \leqq \rho_0$ and $\sigma \leqq \rho_k \leqq 2\rho_0 - \sigma$. Set $x_{k+1} = P(x_k - \rho_k \nabla f(x_k))$. Then:*

(i) *The sequence x_k belongs to S, $(x_{k+1} - x_k)$ converges to 0, and $f(x_k)$ converges downward to a limit L.*

(ii) *If S is compact, z is a cluster point of $\{x_k\}$, and ∇f is continuous in some neighborhood of z, then z is a stationary point. If z is unique, x_k converges to z, and z minimizes f on C.*

(iii) *If S is convex and $f''(x, h, h) \geqq \mu\|h\|^2$ for each $x \in S$, $h \in H$ and some $\mu \geqq 0$, then $L = \inf\{f(x): x \in C\}$.*

(iv) *Assume* (iii) *with S bounded. Weak cluster points of $\{x_k\}$ minimize f on C.*

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