ON THE GALOIS THEORY OF INSEPARABLE EXTENSIONS

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Let K be a field of characteristic $p \neq 0$ and Der K denote the set of all derivations of K into itself, i.e., of additive maps ϕ of K into itself such that $\phi(xy) = \phi(x)y + x\phi(y)$ for all x and y in K. Then Der K is (1) a vector space over K, (2) closed under the formation of pth powers, i.e., ϕ in Der K implies ϕ^p is in Der K, and (3) a Lie subring of the ring of additive endomorphisms of K, i.e., ϕ, ψ in Der K imply $\phi\psi - \psi\phi$ is in Der K. A theorem of Jacobson [6] gives a relationship between the subfields k of K with $K^p \subset k$ and $[K:k] < \infty$, and "restricted" Lie subrings of Der K, which are finite-dimensional vector spaces over K, i.e., the subsets D satisfying (1), (2), and (3) with $\dim_K D < \infty$. Indeed, given such a k, then the set $\mathrm{Der}_k K$ of those derivations of K into itself which vanish on k is clearly a restricted Lie subring of finite dimension over K, and if $[K:k] = p^m$, then one has $\dim_K(\mathrm{Der}_k K) = m$. Conversely, Jacobson demonstrated that given a restricted subring D of Der K which is a finite-dimensional vector space over K, and denoting by k the constant field of D, i.e., the set of all x in K such that $\phi(x) = 0$ for all ϕ in D, then in fact $D = \mathrm{Der}_k K$, whence if $\dim_K D = m$, then $[K:k] = p^m$. If ϕ is in $\mathrm{Der}_k K$ then we say that ϕ is a derivation "over k."

It is remarkable that from the hypotheses of Jacobson's theorem *one may delete the assumption that D be a Lie subring* of Der K. In fact, if we define a *restricted subspace* of Der K to be a subset which is a vector space over K and which is closed under the formation of pth powers, then one may assert: *If D is a finite-dimensional restricted subspace of* Der K, *and if k is the field consisting of all x in K such that $\phi(x) = 0$ for all ϕ in D, then $D = \mathrm{Der}_k K$.* It follows a posteriori that D must be a Lie subring of Der K. The purpose of the present note is to give a simple proof of this strengthened result. For connections with other work, see the "concluding remarks."

1. **Derivations of a field.** Let K be a field of characteristic $p \neq 0$ and Der K denote the set of derivations of K into itself. Given ϕ in Der K, the set of x in K such that $\phi(x) = 0$ forms a subfield K_ϕ of K called the constant field of ϕ; if x is in K_ϕ then $\phi(xy) = x\phi(y)$ for all y in K, and conversely. We note that since $\phi(x^p) = 0$ for all x in K, we

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