DEMICONTINUITY, HEMICONTINUITY AND MONOTONICITY

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Recently the notions of monotone, demicontinuous and hemicontinuous functions have been introduced in connection with nonlinear problems in functional analysis (Browder [1; 2; 3; 4; 5], Minty [6; 7; 8]). The object of the present note is to show that under rather general conditions, hemicontinuity is equivalent to demicontinuity for monotone functions.

Let X be a (real or complex) Banach space and X^* its adjoint space as the set of all bounded conjugate-linear functionals on X. The value of $f \in X^*$ at $u \in X$ is denoted by (f, u). We use the notations $\rightarrow$ and $\rightharpoonup$ for strong convergence in X (or in X^* or in the set of real numbers) and weak* convergence in X^*, respectively.

Let G be a function from X to X^* with domain $D = D(G) \subset X$. G is said to be *demicontinuous* if $u_n \in D$, $n = 1, 2, 3, \cdots$, $u \in D$ and $u_n \rightarrow u$ imply $Gu_n \rightharpoonup Gu$. G is *hemicontinuous* if $u \in D$, $v \in X$ and $u + t_n v \in D$, where t_n is a sequence of positive numbers such that $t_n \rightarrow 0$, imply $G(u + t_n v) \rightharpoonup Gu$. We shall say that G is *locally bounded* if $u_n \in D$, $u \in D$ and $u_n \rightarrow u$ imply that Gu_n is bounded. Obviously a demicontinuous function is hemicontinuous and locally bounded.

G is said to be *monotone* if $\mathrm{Re}(Gu - Gv, u - v) \geqq 0$ for $u, v \in D$.

These definitions may be void if D is too arbitrary. In what follows we shall assume that D is *quasi-dense*. By this we mean that for each $u \in D$ there is a dense subset M_u of X such that for each $v \in M_u$, $u + tv \in D$ for sufficiently small $t > 0$ (the smallness of t depending on v). Thus any open subset of X as well as any dense linear manifold of X is quasi-dense.

THEOREM 1. *Let G be a monotone function from X to X^* with a quasi-dense domain D. Then G is demicontinuous if and only if it is hemicontinuous and locally bounded.*

PROOF. By the remark given above, it suffices to prove the "if" part. Suppose G is hemicontinuous and locally bounded. Let $u_n \rightarrow u$, $u_n, u \in D$. We have to show that $Gu_n \rightharpoonup Gu$. Obviously we may assume that $u_n \neq u$.

Let M_u be the dense subset of X used in the definition of D being quasi-dense. Let $v \in M_u$ and $t_n = \|u_n - u\|^{1/2}$. Then $t_n > 0$, $t_n \rightarrow 0$, $w_n = u + t_n v \in D$ for sufficiently large n and