RECURRENT RANDOM WALKS WITH ARBITRARILY LARGE STEPS

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Introduction. The random walk generated by the distribution function (d.f.), F, is the sequence $S_n = X_1 + \cdots + X_n$, of sums of independent and F-distributed random variables. If $P\{|S_n| < 1, \text{i.o.}\} = 1$, F is called recurrent.[1] If F is not recurrent, $P\{|S_n| \to \infty\} = 1$ [1], and F is called transient. This note contains a proof that there are recurrent distributions with arbitrarily large tails. This assertion was made without proof in [2], where it is shown that for convex distributions, such behavior cannot take place.

1. **Comparing random walks.** We shall prove the following theorem.

THEOREM. *If $\epsilon = \epsilon(x)$ is defined for $x \geqq 0$, and $\epsilon(x) \to 0$, as $x \to \infty$, then there is a* recurrent *distribution function F, for which, for some x_0,*

$$(1.1) \qquad 1 - F(x) = F(-x) \geqq \epsilon(x), \qquad x \geqq x_0.$$

This result may be restated in the following way. For any d.f. G, there is a recurrent d.f. F, and a sample space W on which sequences $X_n = X_n(w)$, $Y_n = Y_n(w)$, $n = 1, 2, \cdots$, may be defined so that for each $w \in W$,

$$(1.2) \quad |Y_n(w)| < |X_n(w)|, \ \text{sign}\, Y_n(w) = \text{sign} X_n(w), \ n = 1, 2, \cdots,$$

where Y_n, $n = 1, 2, \cdots$, are independently G-distributed, and X_n, $n = 1, 2, \cdots$, are independently F-distributed. Considering G transient, we have

$$(1.3) \quad P\{|Y_1 + \cdots + Y_n| \to \infty, \ |X_1 + \cdots + X_n| < 1, \text{i.o.}\} = 1$$

We remark that F cannot be chosen convex. If F is (eventually) convex, and $1 - F(x) = F(-x) \geqq 1 - G(x) = G(-x)$, where G is transient, then F is also transient [2].

The idea of the proof of the theorem is to move out the mass of G and bunch it up, leaving large gaps, so that the remaining steps somehow cancel themselves out.

2. **Proof of the cancellation theorem.** For symmetric F, the condition that F be recurrent is a tail condition [2], and may be stated

[1] i.o. or infinitely often here means for infinitely many $n = 1, 2, \cdots$.