NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS[1]

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1. **Introduction.** Let G be a locally compact group, let $C_0(G)$ be the Banach space of all continuous complex-valued functions on G which vanish at infinity, and let $M(G)=C_0(G)^*$. In [2] Glicksberg has identified all subgroups of the unit ball $\Sigma_{M(G)}$ in $M(G)$ for locally compact abelian groups G, and all positive subgroups of $\Sigma_{M(G)}$ when G is a compact group. In a subsequent paper [3], he used these results to give the structure of norm decreasing homomorphisms ϕ: $L^1(F)\to M(G)$ in the situations when (1) F is locally compact and G is locally compact *abelian*, or (2) F is locally compact, G is *compact*, and $\mu \geqq 0 \Rightarrow \phi(\mu)\geqq 0$ for $\mu \in L^1(F)$. The results given here identify all subgroups of $\Sigma_{M(G)}$ for any locally compact group G, and give the structure of all norm decreasing homomorphisms ϕ: $L^1(F)\to M(G)$ where F and G are locally compact groups. At the same time we show that every homomorphism of this type has a natural extension to a norm decreasing homomorphism ϕ: $M(F)\to M(G)$.

2. **Subgroups of the unit ball in $M(G)$.** The structure theorems are based on the following observations about measures in $M(G)$. If $\mu \in M(G)$ we let $s(\mu)$ denote the support of μ and let $|\mu|$ denote the total variation of μ [1, p. 122]. The norm of $\mu \in M(G)$ is indicated by $\|\mu\|$.

THEOREM 1. *If G is a locally compact group and if $\mu, \lambda \in M(G)$ are such that $\|\mu * \lambda\| = \|\mu\|\cdot\|\lambda\|$, then*
(1) $s(\mu*\lambda)=(s(\mu)s(\lambda))^-$,
(2) $|\mu*\lambda| = |\mu| * |\lambda|$.

In Glicksberg [2], similar results are proved in limited situations: (1) is proved for positive measures, and (2) is proved for measures in a subgroup of $\Sigma_{M(G)}$ when G is compact.

If K is a compact subgroup of a locally compact group G, we denote normalized Haar measure on K as m_K. If $K^\wedge$ is the collection of continuous unimodular multiplicative functions on K and if $\rho \in K^\wedge$, then the measure ρm_K is defined such that $\int_G \psi d\rho m_K = \int_K \psi\cdot\rho dm_K$ for $\psi \in C_0(G)$.

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