ON BOUNDING HARMONIC FUNCTIONS BY LINEAR INTERPOLATION

BY H. F. WEINBERGER[1]

Communicated by E. Calabi, February 10, 1964

It is well known [1], [4] that Poisson's formula for the value at the origin O of a function which is harmonic inside a circle $(x-x_0)^2+(y-y_0)^2=A^2$ can be written in the form

$$u(O)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta+\pi)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\,d\theta,$$

where $r=R(\theta)$ is the polar equation of the boundary. Thus the value of a harmonic function at any point in a circle is an average of the values obtained by linear interpolation of the boundary values at the ends of each chord through the point.

In particular, it follows that

$$u(O)\leqq\max\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\cdot$$

It is tempting to conjecture that a similar inequality holds for harmonic functions in any convex or even star-shaped domain. Recently J. Barta [2], [3] has given two incomplete proofs of this conjecture.

We shall show that in general no inequality of the form

$$u(O)\leqq M\max\frac{R(\theta+\pi)u(R(\theta),\theta)+R(\theta)u(R(\theta+\pi),\theta+\pi)}{R(\theta)+R(\theta+\pi)}\tag{1}$$

can hold for all harmonic functions in a star-shaped domain $r<R(\theta)$. In fact, an inequality of the form (1) holds for each point O of a convex domain D only if D is the interior of a circle.

We first prove:

LEMMA. *Let G be the Green's function with singularity at O for the two-dimensional domain D: $r<R(\theta)$. An inequality of the form* (1) *holds for all harmonic functions u if and only if the identity*

$$\begin{aligned}&R(\theta)(R(\theta)^2+R'(\theta)^2)^{1/2}\frac{\partial G}{\partial n}(R(\theta),\theta)\\&\quad=R(\theta+\pi)(R(\theta+\pi)^2+R'(\theta+\pi)^2)^{1/2}\frac{\partial G}{\partial n}(R(\theta+\pi),\theta+\pi)\end{aligned}\tag{2}$$

---

[1] This research was supported by NSF Grant No. GP-2280.