A COHOMOLOGY THEORY BASED UPON SELF-CONJUGACIES OF COMPLEX VECTOR BUNDLES[1]

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Communicated by W. S. Massey, March 3, 1964

Let $\xi: E \to X$ be a complex n-plane bundle. A self-conjugacy of ξ is a map $\chi: E \to E$ which is a norm-preserving bundle equivalence on the underlying $2n$-dimensional real bundle and has the property that $\chi(\lambda e) = \bar{\lambda}\chi(e)$ for $e \in E$ and λ complex.

If χ_i is a self-conjugacy of ξ_i, define (ξ_1, χ_1) to be equivalent to (ξ_2, χ_2) if there is a bundle equivalence $\mu: E_1 \to E_2$ with $\mu^{-1}\chi_2\mu$ homotopic to χ_1; i.e. there is a continuous map $H: E_1 \times I \to E_1$ with $H|E_1 \times 0 = \mu^{-1}\chi_2\mu$, $H|E_1 \times 1 = \chi_1$, and $H|E_1 \times t$ is a self-conjugacy of ξ_1 for each $t \in I$.

If χ is a self-conjugacy of ξ, χ may be interpreted as a cross-section of the bundle ξ' associated to ξ, whose fibre is the space of 1-1 norm-preserving self-conjugacies of the fibre of ξ. The fibre of ξ', denoted $V(n)$, is homeomorphic to $U(n)$, the unitary group, but there is no canonical homeomorphism. A noncanonical homeomorphism is given by left or right composition with any element of $V(n)$. Let $W(n)$ be the total space of the $V(n)$ bundle associated to the universal bundle over $BU(n)$. Then if $f: X \to BU(n)$ is a classifying map for ξ and χ is a self-conjugacy of ξ, χ interpreted as a cross-section of ξ' gives a lifting of f to a map $g: X \to W(n)$. Conversely any such lifting of f gives a self-conjugacy of ξ.

THEOREM I. *This construction defines a* 1-1 *correspondence between homotopy classes of maps* $X \to W(n)$ *and equivalence classes of bundles with self-conjugacy with* X *as base space.*

There is no difficulty about extending Whitney sums to bundles with self-conjugacy. One writes:

$$(\chi_1 \oplus \chi_2)(e_1, e_2) = (\chi_1 e_1, \chi_2 e_2)$$

verifying easily that if χ_i is a self-conjugacy of ξ_i, then $\chi_1 \oplus \chi_2$ is a self-conjugacy of $\xi_1 \oplus \xi_2$.

The definition of Whitney sum gives rise to a definition of stable equivalence: two bundles with self-conjugacy are stably equivalent if their sums with some third bundle with self-conjugacy are equivalent. A classifying space for stable equivalence classes is $Z \times W$, where Z

[1] While engaged in the research for this paper, the author was supported by NSF contract GP-2037.