ON ORDINALS

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Ordinals in von Neumann's sense would naturally be defined by the transfinite recursion:

(1) $NO(x)$ for $(y)(S``y \subseteq y \cdot (z)(z \subseteq y \cdot\supset\cdot \bigcup z \in y) \cdot\supset\cdot x \in y)$

were it not for the excessive demands on class existence for the values of 'y'.

A trick of inversion (see [1]) obviated the need of infinite values of 'y' in the definition of natural numbers: give the role of y to its complement $\bar{y}$ (without assuming existence of $\bar{y}$) and then reduce. I.e., put bars over the occurrences of 'y' after the quantifier and reduce. The same trick on (1) gives

(2) $NO(x)$ for $(y)(x \in y \cdot \breve{S}``y \subseteq y \cdot\supset (\exists z)(\bigcup z \in y \cdot y \cap z = \Lambda))$.

(2) and (1) are equivalent for naive set theory, since, taking y as $\bar{y}$ in either, you get the other. But the superiority of (2) is that it requires, for each x, no y bigger than $S`x$.

Using the axioms of power set and Aussonderung, we can prove the law of transfinite induction:

(3) $(x)(Fx \supset F(S`x)) \cdot (y)(y \subseteq F \cdot\supset F(\bigcup y)) \cdot NO(z) \cdot\supset Fz.$

PROOF. By power set and Aussonderung, we can take y in (2) as $\{x: x \subseteq z \cdot \sim Fx\}$; so, by the last premise,

$$z \subseteq z \cdot \sim Fz \cdot (x)(S`x \subseteq z \cdot \sim F(S`x) \cdot\supset\cdot x \subseteq z \cdot \sim Fx) \cdot\supset (\exists y)(\bigcup y \subseteq z \cdot \sim F(\bigcup y) \cdot \sim(\exists x)(x \subseteq z \cdot x \in y \cdot \sim Fx)).$$

Dropping '$z \subseteq z$' as true and '$x \subseteq z$' as implied by '$S`x \subseteq z$', and contraposing, we have:

$$(x)(Fx \cdot S`x \subseteq z \cdot\supset F(S`x)) \cdot (y)(\bigcup y \subseteq z \cdot (x)(x \subseteq z \cdot x \in y \cdot\supset Fx) \cdot\supset F(\bigcup y)) \cdot\supset Fz.$$

But '$x \subseteq z$' is redundant in view of '$\bigcup y \subseteq z$' and '$x \in y$'. Dropping it, we see that the two clauses of the antecedent here follow respectively from the second and third premises of (3). So Fz, q.e.d.

At this point we can apply Theorem I of [2], according to which, if a system $\mathfrak{S}$ contains extensionality, Aussonderung, and self-