ON THE REAL PARTS OF ZEROS OF EXPONENTIAL POLYNOMIALS

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Communicated by P. E. Connor, October 25, 1963

In the study of linear differential systems with a single time lag we are led to the study of the nature of the zeros of the characteristic exponential polynomial, which are in general infinite in number. In questions of stability the primary problem is determining the nature of the real parts of these zeros, stability only occurring if the real parts are all negative. The most notable achievement in this line is that of Pontrjagin [2] which states that if the exponential polynomial lacks a principal term (see Bellman and Cooke [1, p. 440]), it has an infinite number of zeros with arbitrarily large positive real parts. What happens when there is a principal term has until now not been discussed. This paper proposes to fulfill that need.

In many applications it is possible to vary the coefficients slightly, and, in so doing, alter the zeros slightly. With only a finite number of zeros with positive real parts it may be possible to alter the coefficients in such a way as to achieve stability. If an infinite number of zeros with arbitrarily large positive real parts occurs, any hope of stabilizing the system vanishes.

We will consider a linear differential system with time lag τ having as its characteristic equation

$$F(z) = z^n + az^{n-1} + \cdots - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots) = 0 \tag{1}$$

where $\tau > 0$, $K \geqq 0$ and $\theta \geqq 0$ are real constants and a and b are complex constants.

THEOREM I. *If $n > m$: the number of zeros of $F(z)$ with positive real part (or lying in any right half plane) is finite; if $K \neq 0$, $F(z)$ has an infinite number of zeros with arbitrarily large negative real part.*

II. *If $n = m$: when $K \neq 0$ $F(z)$ has an infinite number of zeros given by*

$$(1/\tau)(\log K + i(\theta + 2k\pi)) + o(1) \tag{2}$$

as $k = 0, \pm 1, \pm 2, \cdots$, and only a finite number of other zeros. If $K < 1$, $F(z)$ has only a finite number of zeros with positive real part. If $K > 1$, $F(z)$ has only a finite number of zeros with negative real part.

III. *If $n < m$: the number of zeros of $F(z)$ with negative real part (or lying in any left half plane) is finite; if $K \neq 0$, $F(z)$ has an infinite number of zeros with arbitrarily large positive real parts.*