AN OBSTRUCTION TO FINITENESS OF CW-COMPLEXES

BY C. T. C. WALL

Communicated by W. S. Massey, November 4, 1963

A cell structure is a convenient means of describing a space; thus it is important to reduce such a structure to a simpler one when possible. For example, it remains unsolved whether a compact topological manifold (or more generally, ANR) has the homotopy type of a finite CW-complex. According to Milnor [2], this would follow from the conjecture that any CW-complex which is dominated by a finite complex has the homotopy type of a finite complex, but we show below that this is false.

Let X be a connected CW-complex, with universal cover $\tilde{X}$, and fundamental group π with (integral) group ring Λ. Consider the following conditions:

(i) X is dominated by a complex of finite type (i.e., one with a finite number of cells of each dimension),

(ii) π and all $H_i(\tilde{X})$ are countable,

(iii)$_N$ For $N<i$, $H_i(\tilde{X})=0$ and $H^i(X;\mathcal{B})=0$ for all coefficient bundles $\mathcal{B}$ (in the sense of Steenrod; generalised to non-abelian coefficients if $i=2$).

Our results are as follows:

(A) If (i) holds, X is homotopy equivalent to a complex of finite type.

(B) If Λ is noetherian, (i) is equivalent to: π is finitely presented, and all $H_i(\tilde{X})$ are finitely generated Λ-modules.

(C) If X is dominated by a countable complex, it is homotopy equivalent to one; this condition is equivalent to (ii).

(E) If (iii)$_N$ holds, and $N\neq 2$, X has the homotopy type of an N-dimensional complex, countable if (ii) holds.

(F) X is dominated by a finite complex if and only if (i) and some (iii)$_N$ hold. When this is the case, and $N\geqq 2$, there is an obstruction $\theta(X)$ in the projective class group $\tilde{K}^0(\Lambda)$, which depends only on the homotopy type of X, and is zero for X finite. If $\theta(X)=0$, X has the homotopy type of a finite complex of dimension $\max(3, N)$. For $N\geqq 2$, any finite complex K of dimension N, and $\alpha\in\tilde{K}^0(\pi_1(K))$, there is a complex X, with the $(N-1)$-type of K, satisfying (i) and (iii)$_N$, and with $\theta(X)=\alpha$.

The proofs are mostly by induction; we obtain complexes K^r and r-connected maps $\phi: K\to X$, where K is finite in (A), countable in (C). We then prove that $\pi_{r+1}(\phi)$ is finitely generated (over Λ) in (A),