A COMPLETE CLASSIFICATION OF THE Δ_2^1-FUNCTIONS

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Suslin has shown that a set is a Borel set if and only if both it and its complement are analytic sets [14]. Kleene has proved an analogous theorem for the hyperarithmetical sets [7; 9]. Those hierarchies are so naturally constructed that we can establish significant propositions with the aid of them. A lot of effort was made to construct a natural hierarchy for B_2-sets. They are, however, incomplete and contain only a small portion of B_2-sets [10]. The situation was the same for the Δ_2^1-functions of the natural numbers[1] and, if we consider the reason why our trials failed [13; 18], we should say that some new principles were required to settle our problem. Shoenfield [20] constructed for the first time a complete hierarchical classification of the Δ_2^1-functions. Namely, he showed, by the aid of the effective version of the uniformization principle of Kondô [11; 1], that every Δ_2^1-function is constructible from a Δ_2^1-ordinal and conversely. Ours has the same character as his in the use of the uniformization principle. We shall define another classification and shall prove it to be complete by using that principle.[2] We shall study our classification in relation to the hyperdegree of Kleene and shall prove that it is neither fine nor coarse. Although we have not done so, comparison of the two complete classifications may be worthy of study.

CLASSIFICATION. Let γ be the unique solution of the condition $(\alpha)(Ex)P(\beta, \bar{\alpha}(x))$.[3] We shall then say that γ is defined by the sieve P and P is a sieve for γ. Let us denote by $\mathfrak{U}$ the set of functions γ defined by recursive sieves. Let $T_{P,\beta}$ be the set of sequence numbers in $\overline{P}^{\langle\beta\rangle}$ which are neither secured nor past secured [8]. For any recursive sieve R, there is a recursive sieve Q for which the identity $T_{R,\beta}=T_{Q,\beta}=\overline{Q}^{\langle\beta\rangle}$ holds for every β. For γ in $\mathfrak{U}$ we shall denote by $\tau(\gamma)$ the smallest of the ordinals $\tau(T_{R,\gamma})$ where R are recursive sieves for γ. $\mathfrak{V}$ is the set of $\tau(\gamma)$ for γ in $\mathfrak{U}$. If γ is in $\mathfrak{U}$, γ is evidently a

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[1] A problem of Tugué [24, p. 117] was negatively solved by him and us. It was also solved by Shoenfield [21] and Gandy [5].

[2] Theorem 1 is a precise formulation of a statement of Kondô's. See our Remark to Theorem 1.

[3] Notations are those of [6; 7; 8; 9]. Some notations are also borrowed from [11]. We shall use Σ_n^t, Π_n^t notation of [1]. Δ_n^t is the intersection of the Σ_n^t and Π_n^t families [20]. Following notations are used: $\langle x_0, \cdots, x_n\rangle$ for $p_0^{x_0} * \cdots * p_n^{x_n}$, $P\langle\mathfrak{a}\rangle$ for the set of sequence numbers u for which $P(\mathfrak{a}, u)$.