NEW AND OLD PROBLEMS FOR ENTIRE FUNCTIONS

LOUIS DE BRANGES

An entire function is a function which is defined and differentiable in the complex plane. Although there is an extensive classical theory of such functions, they are now best known for their occurrence in the theory of integral transformations. The first problem is therefore to obtain integral representations of entire functions.

The representation theory of entire functions as L^2 Fourier transforms is due to Paley and Wiener [18]. Consider the Fourier representation of a function

$$F(x) = \int f(t)e^{ixt}dt$$

in L^2 when its Fourier transform $f(x)$ vanishes outside of some finite interval $[-a, a]$. In this case, if $F(x)$ is suitably redefined in a set of measure zero, it is the restriction to the real axis of an entire function $F(z)$ which satisfies the inequality

$$|F(x+iy)|^2 \leqq \int |F(t)|^2 dt(e^{2ay} - e^{-2ay})/(4\pi y)$$

in the complex plane. Paley and Wiener show that an entire function which satisfies this inequality and is square integrable on the real axis is the Fourier transform of an L^2 function which vanishes outside of $[-a, a]$. Actually they show more, but this is a technical point for which I refer to Boas's book on entire functions [2].

The significance of the theorem is that it reduces the study of the L^2 Fourier transformation to the study of entire functions which are square integrable on the real axis. Corresponding to any given number $a > 0$, there is a Hilbert space of entire functions, characterized by an inequality in the complex plane. These Paley-Wiener spaces are contained isometrically in L^2. They are totally ordered by inclusion. Their union is dense in L^2 and their intersection contains no nonzero element. A knowledge of these spaces is sufficient to determine the Fourier transformation. The problem now is to obtain a similar characterization of functions which are representable for other integral transformations.

An address delivered before the Madison meeting of the Society on November 30, 1963 by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors November 30, 1963.