DEFORMATIONS OF RIEMANNIAN STRUCTURES

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It has been shown in [1] that a Kählerian deformation of an irreducible compact Hermitian symmetric space V is again symmetric and is isomorphic to it. The aim of this note is to prove this for any compact Hermitian symmetric space, irreducible or not. We first prove the following Theorem 1 which is a slight generalization[1] of Theorem 2 in [1].

THEOREM 1. *Let V and B be two differentiable manifolds and let $R(t)$ be a riemannian structure on V which depends in a differentiable way on $t\in B$; let F be any compact riemannian manifold. Then the set of points $t\in B$ such that $(V, R(t))$ is isomorphic to F is closed in the set of points $t\in B$ such that $(V, R(t))$ is complete.*

PROOF. Denote by C the set of all points $t\in B$ such that $V_t=(V, R(t))$ is complete and let t_n be a sequence of points in C converging to a point t_0 such that (i) $t_0\in C$, and (ii) $V_{t_n}=(V, R(t_n))$ is isomorphic to F for all $n>0$. We prove that V_0 is isomorphic to F. Let h_n be an isomorphism of F onto $V_{t_n}=(V, R(t_n))$ and let r_0 be an orthonormal frame in the tangent space $T_0(F)$ at p_0 of F; denote by r_{t_n} the orthonormal frame $h_n(r_0)$ at $x_n=h_n(p_0)$ in V_{t_n}. Since V_0 is compact, the sequence of points $\{x_n\}$ has a limit point x_0. Since the set of all orthonormal frames at all points of a compact neighbourhood of x_0 in V_0 is again compact, the sequence $\{r_{t_n}\}$ admits a limit point r_0'. Let l_t be the linear mapping of $T_{x_1}(V_t)$ onto $T_{x_n}(V_{t_n})$ which maps r_1 onto r_{t_n}; let u be a unit vector at x_1 and let y be the end-point of the geodesic arc of length s, of origin x_1 and tangent to u; then $y=\phi(s, u)$ where ϕ is a differentiable function of u and of s. If y_t is the end-point of the geodesic arc of origin x_t and tangent to $l_t(u)$, we have $y_t=\phi(s, u, t)$ where $\phi(s, u, 1)=\phi(s, u)$. For $t>0$, we have $y_t=h_t(y)$; as $t_n\to 0$, $h_n(u)$ tends to a vector $h_0(u)$ and $h_n(y)$ tends to a point $h_0(y)=\phi(s, u, 0)$. It is easy to see that h_0 is a well-defined differentiable mapping of V_1 into V_0; since V_0 is complete, it follows that h_0 is onto. We prove that h_0 is one-to-one; let z_1 and z_2 be two distinct points of V_1 and let d_t denote the metric defined by $R(t)$ on V_t. Since

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