A GENERALIZED MORSE THEORY

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1. **Abstract theory.** Let M be a C^2-Riemannian manifold without boundary modeled on a separable Hilbert space (see Lang [3]). For $p \in M$ we denote by $\langle\ ,\ \rangle_p$ the inner product in the tangent space M_p and we define a function $\|\ \|$ on the tangent bundle $T(M)$ by $\|v\| = \langle v, v\rangle_p^{1/2}$ for $v \in M_p$. Given p and q in the same component of M we define $\rho(p, q) = \operatorname{Inf}\int_0^1 \|\sigma'(t)\| dt$, where the Inf is over all C^1 paths $\sigma: [0, 1] \to M$ such that $\sigma(0) = p$ and $\sigma(1) = q$. Just as in the finite dimensional case one shows that ρ is a metric on each component of M which is consistent with the manifold topology. If each component of M is complete in this metric M is called a complete Riemannian manifold and *we assume this in all that follows.* Let $f: M \to R$ be a C^2 function. Then df, the differential of f, is a C^1 cross section of the cotangent bundle of M, hence there is a uniquely determined C^1 vector field ∇f on M, the gradient of f, such that $df_p(v) = \langle v, \nabla f(p)\rangle_p$ for $v \in M_p$. We denote by ϕ_t the maximum local one-parameter group generated by $-\nabla f$. A critical point of f is a point where ∇f vanishes; equivalently a stationary point of ϕ_t. At a critical point p of f there is a uniquely determined continuous bilinear form $H(f)_p$ on M_p, the Hessian of f at p, such that $H(f)_p(u, v) = d^2(f \circ \phi^{-1})(d\phi_p(u), d\phi_p(v))$ if ϕ is any chart at p. The supremum of the dimensions of subspaces on which $H(f)_p$ is negative (positive) definite is called the index (coindex) of f at p. $H(f)_p$ is symmetric, hence there is a uniquely determined bounded self-adjoint operator A on M_p such that $H(f)_p(u, v) = \langle Au, v\rangle_p$. The critical point p is called nondegenerate if A has a bounded inverse. In this case p is isolated in the set of critical points.

Let $f^{a,b} = f^{-1}[a, b]$ and $f^b = f^{-\infty,b}$. Morse theory is concerned with relating the structure of the critical point set of f in $f^{a,b}$ with the homology, homotopy, homeomorphism, and diffeomorphism type of the pair (f^b, f^a). We shall be concerned with the Morse theory of pairs (M, f) as above which satisfy at least the following extra condition:

(C) If S is a subset of M on which $|f|$ is bounded but on which $\|\nabla f\|$ is not bounded away from zero, then there is a critical point of f in the closure of S.

Note that if f is proper (which implies that M is finite dimensional) and in particular if M is compact then condition (C) is automatically satisfied. More interesting though is the fact, which we will make