THE COHOMOLOGY OF GROUP EXTENSIONS

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Communicated by Deane Montgomery, May 21, 1963

1. **Introduction.** Suppose Π is a group with a finitely generated abelian normal subgroup M and let $\Phi=\Pi/M$, i.e. Π satisfies the exact sequence

$$(*) \qquad 0 \to M \to \Pi \to \Phi \to 1.$$

The isomorphism class of Π is determined by (A) the groups M and Φ, (B) the structure of M as a Φ-module, and (C) a cohomology class $a \in H^2(\Phi; M)$ which describes the extension (cf. [1]). In principle then it should be possible to compute $H^*(\Pi)$, the cohomology ring of Π, from the above information. Practically, however, this seems to be impossible in general even if we assume known the cohomology of M and Φ. Our objective here is to solve an approximation to this problem.

The Hochschild-Serre spectral sequence [2] provides us with a sequence of differential rings (E_r, d_r) $(r=1, 2, \cdots)$ which approximate the ring $H^*(\Pi)$ and such that $E_{r+1}=H(E_r, d_r)$. Hochschild and Serre computed E_2 and found that $E_2^{p,q} \cong H^p(\Phi; H^q(M))$. So E_2 depends only on (A) and (B) and is therefore a rather crude approximation to $H^*(\Pi)$. We determine d_2 (and hence E_3) in terms of (A), (B), and (C). Hochschild and Serre found d_2 on $E_2^{*,1}$ ("the first row"), and our results can be thought of as a generalization of theirs. We assume we have coefficients in a field F although the results are valid in somewhat greater generality.

In §2 we generalize a technique in [2] and define two new spectral sequences $\hat{E}_r$ and $\overline{E}_r$ and a cup product pairing from $\hat{E}_r \otimes \overline{E}_r$ to E_r. The problem of computing d_2 in E_2 is reduced to computing $\hat{d}_2$ on a sequence of classes $f^n \in \hat{E}_2^{n,0}$, and then the value of d_2 on a class in $E_2^{n,p}$ is equal to the cup product of $b^n=\hat{d}_2(f^n)$ and an appropriate class in $\overline{E}_2^{0,p}$.

In §3 we assume that (*) splits or equivalently that $a=0$. In this case the entire spectral sequence $(\mathfrak{E}_r, \mathfrak{d}_r)$ depends only on (A) and (B). The classes $v^n=\mathfrak{d}_2(f^n)$ obtained in this case are called characteristic classes of the Φ-module M. They provide some measure of the difference between the cohomology of the split extension $\Phi \cdot M$ and that of the direct product $\Phi \times M$.

[1] Work supported by Air Force Office of Scientific Research.

[2] Work supported by a National Science Foundation Postdoctoral Fellowship.