UNIFORM CROSS NORMS AND TENSOR PRODUCTS OF BANACH ALGEBRAS[1]

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1. **Introduction.** The present work grew out of an attempt to answer three questions. The first two were raised by B. R. Gelbaum [1]. One is: Which cross norms α of the algebraic tensor product $A \otimes B$ of two Banach algebras A and B are compatible with multiplication? Compatibility with multiplication means that

$$\alpha(t_1 t_2) \leqq \alpha(t_1)\alpha(t_2), \tag{1.1}$$

for every two tensors $t_1, t_2 \in A \otimes B$. The second question is: Is the so-called least cross norm λ, in particular, compatible with multiplication? The third question was raised by B. R. Gelbaum and the author in [2]. It is: Given two Banach spaces E and F each of which has a Schauder basis, for what cross norms α do the resulting complete tensor products $E \otimes_\alpha F$ have Schauder bases?

The present state of knowledge concerning the first two questions is as follows. In [3] Gelbaum (see also Tomiyama [4]) defined on the algebraic tensor product $A \otimes B$ of two Banach algebras a multiplication which is the linear extension of the following multiplication on decomposable tensors.

$$(U_1 \otimes V_1)(U_2 \otimes V_2) = U_1 U_2 \otimes V_1 V_2, \tag{1.2}$$

where $U_1, U_2 \in A$ and $V_1, V_2 \in B$. Under this multiplication $A \otimes B$ becomes a complex algebra. The greatest cross norm γ on $A \otimes B$ is compatible with this multiplication, which can be extended to $A \otimes_\gamma B$, turning the latter into a Banach algebra. It is not known, in general, whether the tensor product $A \otimes_\gamma B$ of two semisimple commutative Banach algebras is semisimple, although this is the case for all known concrete cases. No single cross norm, of a general character and known to be different from γ, has been found that is compatible with multiplication. Gelbaum has shown that the nuclear (trace) norm of Grothendieck, which Gelbaum derived independently, is compatible with multiplication, but no single instance is known where that norm differs from γ. In particular, little seems to be known about the compatibility with multiplication of the least cross norm λ of $A \otimes B$.

In the present work we exhibit a broad class of cross norms, includ-

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