HILBERT ALGEBRAS WITH IDENTITY

BY LARS INGELSTAM

Communicated by G. D. Mostow, July 3, 1963

With a Hilbert algebra with identity (HAI) we mean a Hilbert space with inner product (x, y) which is also an associative Banach algebra with identity e; the norm $\|x\| = (x, x)^{1/2}$ satisfying

$$\|xy\| \leqq \|x\| \cdot \|y\|, \tag{1}$$

$$\|e\| = 1. \tag{2}$$

An HAI is called *real* if it is a real Hilbert space and a real algebra; *complex* if it is complex in both respects.

As a consequence of a result on the geometric properties of the unit sphere in Banach algebras, originally due to Bohnenblust and Karlin [**2**], one easily gets

THEOREM 1. *A complex Hilbert algebra with identity is isomorphic to the complex numbers.*

(This is a rephrasing of [**3**, Corollary 2, p. 25].)

In connection with this, it was conjectured by I. Kaplansky[1] that every real HAI must be isomorphic to the reals, complexes or quaternions. The object of this note is to prove that this is true. (Of course if condition (2) or the assumption of identity is dropped there are many other examples.) In particular, we will see that the given conditions imply that the norm must satisfy

$$\|xy\| = \|x\| \cdot \|y\|,$$

in other words be an *absolute value.*

The proof depends partly on techniques developed in [**3**]. We start with two preliminary results.

PROPOSITION 1. *For an element x in a real HAI the conditions*

$$1^\circ \qquad (e, x) = 0,$$

$$2^\circ \qquad \|\exp \alpha x\| = 1 \quad \textit{for all real } \alpha$$

are equivalent.

PROOF. We define

[1] Personal letter, April, 1963. I want to thank Professor Kaplansky for directing my attention to this enjoyable problem.