FUNDAMENTAL POLYGONS[1]

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1. **Siegel's theorem.** The following elegant theorem is due to C. L. Siegel [**3**; **4**]:

THEOREM 1. *Let Θ denote a group of Möbius transformations mapping $\Delta=\{|z|<1\}$ onto itself which is properly discontinuous at each point of Δ. If the common hyperbolic area of the associated Poincaré polygons is finite, then each Poincaré polygon associated with Θ has a finite number of sides, none on $C=\{|z|=1\}$.*

A group Θ satisfying the hypothesis of Siegel is necessarily of the first kind. The question arises whether there is a theorem of the Siegel type for Θ of the second kind. It is one of the objects of the present investigation to establish such a theorem and to draw consequences of this result taken in conjunction with Siegel's theorem. The other object is to study the relation between the parabolic transformations of an unrestricted Θ and the cusps of an associated non-euclidean convex fundamental polygon (n.e.c.f.p.). To be precise, we understand by this term a set Π ($\subset\Delta$) satisfying the following conditions: (1) it is non-euclidean convex, (2) every point of Δ is Θ-equivalent to a point of Π, (3) no two distinct points of int Π are Θ-equivalent, (4) Π is closed in the sense of the topology of Δ, (5) for each point of Δ there exists a neighborhood intersecting $\tau(\Pi)$ for only a finite set of $\tau\in\Theta$, (6) (fr $\Pi)\cap\Delta$ is piecewise hyperbolic rectilinear. The Poincaré polygons are special cases of n.e.c.f.p. We shall refer to n.e.c.f.p. as *fundamental polygons.*

2. **Reformulation of Siegel's theorem in terms of the quotient Riemann surface.** We indicate how the condition of Siegel may be recast in terms independent of fundamental polygons.

If Θ is a group of the first kind, then the component containing Δ of the set of points at which Θ is properly discontinuous is Δ. If Θ is of the second kind, the set of points at which Θ is properly discontinuous is connected and contains Δ properly. Let Ω denote Δ in the first case and the full set of points at which Θ is properly discontinuous in the second case. Let $\phi(z)=\{\tau(z)\mid\tau\in\Theta\}$, i.e. $\phi(z)$ is the *orbit* of a point z with respect to Θ. The image of Ω with respect to the

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