INVARIANT SUBSPACES OF NONSELFADJOINT TRANSFORMATIONS

BY LOUIS DE BRANGES

Communicated by P. R. Halmos, April 17, 1963

This note comments on recent Russian results in Hilbert space. Macaev [9] has given a fundamental estimate of completely continuous transformations which have no nonzero spectrum. The same estimate is true of transformations with imaginary spectrum.

THEOREM I. *Let T be a densely defined transformation in a Hilbert space $\mathcal{H}$ such that T^* has the same domain as T and $T-T^*$ has a completely continuous extension. Suppose that*

$$T - T^* \subset 2i \sum \operatorname{sgn} k\, c_k \bar{c}_k, \tag{1}$$

where (c_k) is an orthogonal set in $\mathcal{H}$, indexed by the odd integers, $\|c_{k+2}\| \leqq \|c_k\|$ for $k>0$, $\|c_{k-2}\| \leqq \|c_k\|$ for $k<0$, and

$$\delta = \sum \|c_k\|^2/|k| < \infty. \tag{2}$$

If the spectrum of T is imaginary, then the spectrum of $\frac{1}{2}(T+T^*)$ is contained in the interval $[-2\delta/\pi, 2\delta/\pi]$.

If a and b are elements of a Hilbert space, $\bar{a}b$ is the inner product, $\bar{a}b = \langle b, a\rangle$, and $a\bar{b}$ is the linear transformation defined by $(a\bar{b})c = a(\bar{b}c)$ for every c in $\mathcal{H}$. The proof of Theorem I is similar to Macaev's except that it depends on the following new estimate of eigenvalues.

THEOREM II. *Let S be an everywhere defined and bounded transformation in a Hilbert space $\mathcal{H}$, which has imaginary spectrum, such that*

$$S - S^* = 2i \sum b_n \bar{b}_n,$$

where (b_n) is an orthogonal set in $\mathcal{H}$ and $\sum \|b_n\|^2$ is finite. Then,

$$S + S^* = 2 \sum \operatorname{sgn} k\, a_k \bar{a}_k,$$

where (a_k) is an orthogonal set in $\mathcal{H}$, indexed by the odd integers, $\|a_{k+2}\| \leqq \|a_k\|$ for $k>0$, $\|a_{k-2}\| \leqq \|a_k\|$ for $k<0$, and

$$\|a_k\|^2 \leqq (2/\pi)\left(\sum \|b_n\|^2\right)/|k|$$

for every k.

Macaev [9] has given a fundamental existence theorem for invariant subspaces. It can be deduced directly from Theorem I without using, as he indicates, an additional estimate of resolvents. Neither