GAPS AT WEIERSTRASS POINTS FOR THE MODULAR GROUP

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Let S be a compact Riemann surface of genus $g \geqq 2$, $h: S \to S$ an automorphism of order N, and H the cyclic group of order N generated by h. One has a representation of H by letting it act on the g complex-dimensional space A_1 of abelian differentials of the first kind on S by $h: \varphi \to \varphi h$ for all $\varphi \in A_1$. At each point $P \in S$ there is a gap sequence $\gamma(P) = \gamma_1(P), \cdots, \gamma_g(P)$ where the $\gamma_j(P)$ are integers satisfying $1 = \gamma_1(P) < \gamma_2(P) < \cdots < \gamma_g(P) < 2g$ such that there is no function on S having a pole of order $\gamma_j(P)$ at P and everywhere else finite. The complementary integers to $\gamma(P)$ in the sequence of integers from 1 to $2g$ are the nongaps at P. A point is a Weierstrass point if $\gamma_g(P) > g$.

In [3] the following was proved:

(I) Suppose $P = h(P)$ is a fixed point for h with gap sequence $\gamma_1, \cdots, \gamma_g$ and that h rotates at P by ϵ, i.e., if z is a local parameter at P, $z(P) = 0$, then $h(z) = \epsilon z + \cdots$, $\epsilon^N = 1$. Then, with respect to a suitable basis for A_1, h is represented by the diagonal matrix $(h) = \text{diag}(\epsilon^{\gamma_1}, \epsilon^{\gamma_2}, \cdots, \epsilon^{\gamma_g})$.

A corollary of this is

(II) If $P = h(P)$ is not a Weierstrass point then h has at most four fixed points. Thus if h has more than four fixed points all its fixed points are Weierstrass points.

Let Γ be the inhomogeneous modular group, $\Gamma(N)$ the principal congruence subgroup of level $N > 2$, $S(N)$ the compactified fundamental domain for $\Gamma(N)$ which is a Riemann surface of genus $g(N) = 1 + N^2(N-6)/24 \prod_{p \mid N} (1 - 1/p^2)$ where the product is over primes dividing N. For details see Chapter 1 of [2]. $\Gamma/\Gamma(N)$ is a group of automorphisms of $S(N)$ whose fixed points are at three kinds of points. Firstly, parabolic points (cusps), equivalent under $\Gamma/\Gamma(N)$ to ∞ which is fixed under the cyclic group of order N generated by (the coset of) $T: \tau \to \tau + 1$. Secondly, elliptic points of order 2, equivalent to $i = \sqrt{-1}$ which is fixed under the cyclic group of order 2 generated by $S: \tau \to -1/\tau$. Thirdly, elliptic points of order 3, equivalent to $\rho = e^{2\pi i/3}$ which is fixed under the cyclic group of order 3 gener-

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