RANDOM DISTRIBUTION FUNCTIONS[1]

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1. **Introduction.** A *random distribution function* $\boldsymbol{F}$ is a measurable map from a probability space $(\Omega, \mathfrak{F}, Q)$ to the space Δ of distribution functions on the closed unit interval I, where Δ is endowed with its natural Borel σ-field, that is, the smallest σ-field containing the customary weak* topology. It determines a *prior* probability measure $P = Q\boldsymbol{F}^{-1}$ in the space Δ. Of course, $\boldsymbol{F}$ is essentially the same as the stochastic process $\{F_t, 0 \leqq t \leqq 1\}$ on $(\Omega, \mathfrak{F}, Q)$, where $F_t(\omega) = \boldsymbol{F}(\omega)(t)$. Therefore, this note can be thought of as dealing with a certain class of random distribution functions, or a class of stochastic processes, or a class of prior probabilities.

Which class? Practically any *base probability* μ on the Borel subsets of the unit square S determines a random distribution function $\boldsymbol{F}$ and so a prior probability P_μ in Δ, which will be described somewhat informally in §2, by explaining how to select a value of $\boldsymbol{F}$, i.e., a distribution function F, at random. §§3, 4 and 5 describe some properties of P_μ. Proofs will be given elsewhere. For ease of exposition, we assume that μ concentrates on, that is, assigns probability 1 to, the interior of S.

2. **The construction.** To select a value F of $\boldsymbol{F}$ at random, begin by selecting a point (x, y) from the interior of S according to μ. The horizontal and vertical lines through (x, y) divide S into four rectangles; consider the closed lower left rectangle L and the upper right one R. The unique (affine) transformation of the form $(u, v) \rightarrow (\alpha u + \beta, \gamma v + \delta)$, α and γ positive, which maps S onto L carries μ into a probability μ_L concentrated on L. The probability μ_R is defined in a similar way. Now select a point (x_L, y_L) at random from the interior of L according to μ_L, and a point (x_R, y_R) at random from the interior of R according to μ_R. As before, (x_L, y_L) determines four subrectangles of L, and (x_R, y_R) determines four subrectangles of R. Consider the lower left subrectangle LL in L, the upper right subrectangle RL in L, and the two analogous subrectangles LR and RR in R. The

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