SINGULAR INTEGRALS AND PARABOLIC EQUATIONS

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1. **Introduction.** A. P. Calderón and A. Zygmund [1; 2] have studied a class of singular integrals, proving that such integrals generate continuous linear transformations of L^p into L^p, $1<p<\infty$. One of the many applications of their results is the derivation of integral estimates for derivatives of solutions of the Poisson equation $\Delta u=f$, where $\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$. Corresponding results have been obtained for a different class of singular integrals; an application of these results is the derivation of integral estimates for derivatives of solutions of the parabolic equation $u_t-\Delta u=f$. We shall briefly outline the development of the singular integrals of Calderón and Zygmund as applied to the equation $\Delta u=f$, and then give the parallel development for the singular integrals associated with the equation $u_t-\Delta u=f$.

2. **The equation** $\Delta u=f$. In n-dimensional Euclidean space R^n let $\Gamma(x)$ be the fundamental solution of Laplace's equation,

$$\Gamma(x) = -\frac{1}{2\pi}\log\frac{1}{|x|}, \qquad n = 2,$$

$$\Gamma(x) = \frac{1}{(2-n)\,\omega_n}|x|^{2-n}, \qquad n > 2,$$

where $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$ and ω_n is the area of the sphere $|x|=1$. Let

$$u(x) = \int_{R^n}\Gamma(x-y)f(y)\,dy \tag{1}$$

where $f\in L^p(R^n)$. Then [1] the second partial derivatives of u exist almost everywhere, and

$$u_{x_ix_j} = \frac{1}{n}\delta_{ij}f(x) + \int_{R^n}k_{ij}(x-y)f(y)\,dy, \tag{2}$$

where $\delta_{ii}=1$, $\delta_{ij}=0$, $i\neq j$, and

$$k_{ij}(x) = \Gamma_{x_ix_j} = \frac{1}{\omega_n}|x|^{-n}\left(\delta_{ij} - n\frac{x_ix_j}{|x|^2}\right).$$

In particular, $\Delta u=f$. The kernel $k=k_{ij}$ has the properties that