A GEOMETRIC METHOD IN DIFFERENTIAL TOPOLOGY

BY ANDREW H. WALLACE

Introduction. My aim here is to explain in an informal way a technique which has proved itself useful in dealing with certain problems in differential topology. The title refers to the fact that the method relies principally on the geometric construction of spherical modifications and certain associated constructions. The general plan of this exposition is, firstly, to define the notion of spherical modification, secondly, to describe some operations which can be performed on these modifications, and thirdly, to give some applications. The article concludes with some remarks on related work. A more formal and detailed presentation of the material treated here appears in [3; 4].

1. **Definition.** Let M_1 be a differentiable manifold of dimension n and let S^r be an r-sphere embedded in M_1. If S^r has a neighborhood in M_1 of the form $S^r \times E^{n-r}$, where E^{n-r} is an $(n-r)$-cell, then S^r is said to be directly embedded in M_1. If, in this case, $S^r \times E^{n-r}$ is removed from M_1, what is left is a manifold having a boundary of the form $S^r \times S^{n-r-1}$. The latter set is, however, also the boundary of $E^{r+1} \times S^{n-r-1}$. Thus, forming the union

$$\{M_1 - (S^r \times E^{n-r})\} \cup (E^{r+1} \times S^{n-r-1})$$

with the appropriate identification of boundaries, a new manifold M_2 is obtained, which can also be made differentiable by a suitable smoothing operation. The manifold M_2 is said to be obtained from M_1 by a spherical modification.

A familiar example of this process is obtained by taking M_1 to be the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$ and M_2 to be the hyperboloid of one sheet $x^2 - y^2 - z^2 = -1$ in Euclidean 3-space. S^r is to be taken as the 0-sphere consisting of the union of the two points $(-1, 0, 0)$ and $(1, 0, 0)$. In this case the set removed is a pair of discs and the set inserted is a circular cylinder.

More generally, M_i could be taken as the quadric hypersurface

$$\sum_{i=1}^{r+1} x_i^2 - \sum_{i=r+2}^{n+1} x_i^2 = 1$$

in $(n+1)$-space, and M_2 as the hypersurface whose equation is obtained from the above by changing the right hand side to -1. In this

An address delivered before the Chicago meeting of the Society on April 13, 1962, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors June 15, 1962.