SOME RESULTS ON INVARIANT THEORY

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1. **Symmetric invariants.** Let V be a finite-dimensional vector space over $\mathbf{R}$. Each $X \in V$ gives rise (by parallel translation) to a vector field on V which we consider as a differential operator $\partial(X)$ on V. The mapping $X \to \partial(X)$ extends to an isomorphism of the complex symmetric algebra $S(V)$ over V onto the algebra of all differential operators on V with constant complex coefficients. Let G be a subgroup of the general linear group $\mathbf{GL}(V)$. Let $I(V)$ denote the set of G-invariants in $S(V)$ and let $I_+(V)$ denote the set of G-invariants without constant term. The group G acts on the dual space V^* of V by

$$(g \cdot v^*)(v) = v^*(g^{-1} \cdot v), \qquad g \in G, v \in V, v^* \in V^*,$$

and we can consider $S(V^*)$, $I(V^*)$, $I_+(V^*)$. An element $p \in S(V^*)$ (a polynomial function on V) is called *G-harmonic* if $\partial(J)p = 0$ for each $J \in I_+(V)$. Let $H(V^*)$ denote the set of G-harmonic polynomial functions.

Let V^C denote the complexification of V. Suppose B is a nondegenerate symmetric bilinear form on $V^C \times V^C$. If $X \in V^C$ let X^* denote the linear form $Y \to B(X, Y)$ on V. The mapping $X \to X^*$ extends to an isomorphism $P \to P^*$ of $S(V)$ onto $S(V^*)$. If G leaves B invariant then $I(V)^* = I(V^*)$.

We shall use the following notation: If E and F are linear subspaces of the associative algebra A then EF denotes the set of all sums $\sum_i e_i f_i$, $(e_i \in E, f_i \in F)$.

THEOREM 1. *Let B be a nondegenerate symmetric bilinear form on $V \times V$ and let G be a Lie subgroup of* $\mathbf{GL}(V)$ *leaving B invariant. Suppose that either* (1) *G is compact and B positive definite or* (2) *G is connected and semisimple. Then*

$$S(V^*) = I(V^*)H(V^*).$$

The case of a compact G was noted independently by B. Kostant. It is a simple consequence of the fact that under the standard strictly positive definite inner product on $S(V^*)$ (invariant under G), the space $H(V^*)$ is the orthogonal complement to the ideal in $S(V^*)$ generated by $I_+(V^*)$. For the noncompact case, let $\mathfrak{g}$ denote the complexification of the Lie algebra of G. It is not difficult to prove that

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