THE OPTIMAL LEBESGUE-RADON-NIKODYM INEQUALITY[1]

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The aim of the present paper is to sketch some further developments of order-integration (cf. [**1**; **2**; **3**]), and in particular to point out how the absence of lattice distributivity introduces some new and interesting aspects of the Lebesgue-Radon-Nikodym Theorem. Details will be published elsewhere.

The formula $\mu(x, y) = v(y) - v(x)$ establishes a 1-1 correspondence between the set of valuations of a lattice L (with identification modulo $v_1 \sim v_2 \Leftrightarrow v_1 - v_2 = \text{const}$) and the set L' of projectivity invariant, additive interval-functions ("quotient-functions") on L. If L is modular, then the equivalence classes of finite chains between x and y ($x \leqq y$) form a directed set in virtue of the Schreier-Ore Theorem. Hence we may define Riemann-Darboux integrals of projectivity invariant interval-functions in the natural way. The R. D. integral of μ is additive (whenever it exists) and will be denoted by S_μ. Now the maximal directed vector subspace $L^* = (L')^+ - (L')^+$ of L' will consist of those $\mu \in L'$ which are of bounded variation in the sense that

$$S_{|\mu|}(x, y) = \sup_{x = x_1 \leqq \cdots \leqq x_n = y} \sum_{i=1}^{n} |\mu(x_{i-1}, x_i)| < \infty; \tag{1}$$

for every interval (x, y). Moreover, L^* will be a conditionally complete vector lattice under the operations $\mu \vee \nu = S_{\mu \vee \nu}$, $\mu \wedge v = S_{\mu \wedge \nu}$. Decomposition of $\mu \in L^*$ in positive and negative parts yields the Jordan decomposition of μ (obtained by G. Birkhoff [**4**]).

The classical (Lebesgue-Vitali) definition of absolute continuity, $\nu \ll \mu$, for functions μ, ν on R can be directly transferred to the case in which μ, ν belongs to the space L^* of some modular lattice L. (The standard definition of $\nu \ll \mu$ for finitely additive measures μ, ν on a Boolean ring is obtained from the general definition by reduction of the chain involved to a two-interval chain by application of the Boolean difference available in this particular case.) The concept of mutual singularity, $\mu \perp \nu$, can be defined for members of L^* in an equally natural way. Let $\mathfrak{A}(\mu)$ denote the closed ideal ("famille com-

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