COMBINATORIAL ASPECTS OF THE ISING MODEL FOR FERROMAGNETISM. II. AN ANALOGUE TO THE WITT IDENTITY

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M. P. Schutzenberger has indicated to the author that the identity [S, p. 208, eqn. 6] used to prove the COROLLARY ON COIN ARRANGEMENTS of [S] is the same as the identity [**H**, p. 170; **W**, p. 156] used to establish the Witt formula for the dimension of the linear space of Lie elements of degree r in a free Lie algebra with k generators over a field of characteristic zero. Since almost no American mathematicians read the *Journal of Mathematical Physics* and no algebraists do, a special case of the main theorem will be presented here without proof and in a form suitable for the algebraist.

First, the Witt identity will be formulated: Let $a_1, \cdots, a_k$ generate a free *semigroup*. Consider only circular words of the same length and period [**H**, p. 170]. Let $M(m_1, m_2, \cdots, m_k)$ be the number of such circular words with m_1 occurrences of a_1, m_2 occurrences of a_2, etc. Let $z_1, \cdots, z_k$ be commuting indeterminates. Then [**H**, p. 170; **W**, p. 156]

$$(1)\quad 1 - z_1 - z_2 - \cdots - z_k = \prod_{m_1, \cdots, m_k \geq 0} (1 - z_1^{m_1} \cdots z_k^{m_k})^{M(m_1, \cdots, m_k)}.$$

From this the Witt formula,

$$M(m_1, \cdots, m_k) = m^{-1} \sum_{d | m_1, \cdots, m_k} \mu(d) \frac{\left(\frac{m}{d}\right)!}{\left(\frac{m_1}{d}\right)! \cdots \left(\frac{m_k}{d}\right)!},$$

where $m = m_1 + \cdots + m_k$, follows by the Möbius Inversion Formula.

With the Witt identity can be associated the following diagram: a collection of k directed planar loops $a_1, \cdots, a_k$ having only one vertex P in common as, for example, in Figure 1. For later purposes it is required that no two loops have a common tangent vector at P. With each circular word w of the kind considered is associated a closed circular path p in a counterclockwise direction (in Figure 1).

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