EXTENSION PROPERTIES OF BANACH SPACES

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A bounded linear transformation a on a Banach space L, to a Banach space M, is called *regular* (see [**9**, Part II, Chapter 2; **2**, pp. 25–26]), in case there is a bounded linear transformation x on M to L, such that $axa=a$. The transformation a is possibly regular only if it is of type IIA [**13**; **14**], that is only if the null space $\mathfrak{N}(a)$ of a has a closed complement $\mathfrak{N}^c$, and if a restricted to $\mathfrak{N}^c$ is an isomorphism g of $\mathfrak{N}^c$ onto the range $\mathfrak{R}(a)$ of a. For such an a, the projection p through $\mathfrak{N}(a)$ onto $\mathfrak{N}^c$ is bounded. Then in order that $axa=a$, it is necessary and sufficient that x be a bounded linear extension to M of the inverse isomorphism g^{-1}. In this case clearly xa is the bounded projection p of L through $\mathfrak{N}(a)$ onto $\mathfrak{N}^c$, and ax is a bounded projection q of M onto $\mathfrak{R}(a)$. Therefore even if a is of type IIA so that the bounded projection p exists, the transformation a cannot be regular if there is no bounded projection q of M onto $\mathfrak{R}(a)$. In case the extension x exists so that a is regular, the transformation $b=px=xax$ on M to L is a *relative inverse* [**2**] of a; that is $aba=a$ and $bab=b$. In case L and M are Hilbert spaces, g is isometric, and p and q are orthogonal projections, the relative inverse b coincides with the adjoint a^* of a. (Every bounded linear transformation on a Hilbert space to a Hilbert space necessarily is of type IA or IIA.[2])

For reasons including those suggested in the preceding paragraph, relations between various properties concerning projections and extensions in Banach spaces are of interest. In this note the equivalence of several such properties, which are seemingly different, is established. The main result is the equivalence of the extension property to the *uniform* extension property. (See the next sections for definitions.) This is a consequence of the following theorem.

THEOREM 1. *If a Banach space B has the property that there is a*

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[2] For suppose that a is bounded on a Hilbert space L to a Hilbert space M. In order that a be of type IB or IIB, it would be necessary that every complement of $\mathfrak{N}(a)$ be not closed, but the orthogonal complement is closed for any $\mathfrak{N}(a)$. In case a is not of type IIA, then neither $\mathfrak{R}(a)$ nor $\mathfrak{R}(a^*)$ is closed, and it follows from $(ax, y)=(x, a^*y)$ that the closure of $\mathfrak{R}(a)$ is the orthogonal complement of $\mathfrak{N}(a^*)$, and that the closure of $\mathfrak{R}(a^*)$ is the orthogonal complement of $\mathfrak{N}(a)$. An a of type IA is not regular, and its adjoint a^* is not a relative inverse of a.