ON VANISHING ALGEBRAS

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Let G be a locally compact group with left invariant Haar measure m. For any measurable subset S of G, define L_S to be that subset of $L^1(G)$ consisting of all functions which vanish (a.e.) on the complement of S. When L_S forms an algebra, we call it a *vanishing algebra*. It is known that when S is a semigroup l.a.e. (i.e., there exists a semigroup T in G such that $S=T$ locally almost everywhere), L_S is a vanishing algebra. The following theorem gives an answer to a problem formulated by A. Simon [**2**]:

THEOREM 1. *Suppose G is unimodular. If L_S is a vanishing algebra and S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

COROLLARY 1. *Suppose G is compact. Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

COROLLARY 2. *Suppose G is abelian and generated by some compact neighborhood of the identity element of G. Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

The proof of Theorem 1 also gives the following more general and involved statement:

THEOREM 2. *Let L_S be a vanishing algebra. Suppose there exists a directed set $\{U_i, i\in I\}$ of symmetric neighborhoods of the identity element e with finite measures, having the property that for almost all the points x of S there exists a $j_x\in I$ such that $m(S\cap xU_i)$ and $m(x^{-1}U_i\cap S^{-1})$ are both $>m(U_i)/2$ as $i\geqq j_x$. Then S is a semigroup l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

THEOREM 3. *If L_S is a self-adjoint vanishing algebra, then S is a group l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a group a.e.*

THEOREM 4. *Let L_S be a vanishing algebra. If S is open, then S is a semigroup l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

THEOREM 5. *If L_S is a maximal vanishing algebra, then S is a closed*

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