ENTIRE FUNCTIONS AND INTEGRAL TRANSFORMS

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If $E(z)$ is an entire function which satisfies

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y>0$ $(z=x+iy)$, let $\mathcal{H}(E)$ be the corresponding Hilbert space of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)/E(t)|^2 dt < \infty$$

and

$$|F(z)|^2 \leqq \|F\|^2 [|E(z)|^2 - |E(\bar{z})|^2]/[2\pi i(\bar{z} - z)]$$

for all complex z. The space is introduced in [**7**], where it is characterized by three axioms. If $E(a, z)$ and $E(b, z)$ are entire functions which satisfy (1), then $\mathcal{H}(E(a))$ will be contained isometrically in $\mathcal{H}(E(b))$ if these functions satisfy the hypotheses of Theorem VII of [**8**]. Isometric inclusions of spaces of entire functions are a basic idea in [**9**] and [**10**]. A fundamental property of these inclusions has only now become available.

THEOREM I. *If $E(a, z)$, $E(b, z)$, and $E(c, z)$ are entire functions which satisfy* (1) *and have no real zeros, and if $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are contained isometrically in $\mathcal{H}(E(c))$, then either $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ or $\mathcal{H}(E(b))$ contains $\mathcal{H}(E(a))$.*

The formal proof depends on techniques of [**2**] and [**3**] for handling difference quotients. To make it precise, one must show that if $f(z)$ and $g(z)$ are entire functions of minimal exponential type such that

$$|yf(z)g(z)| \leqq |f(z)| + |g(z)|$$

for all complex z, then $f(z)g(z)$ vanishes identically. This is proved by a method of Carleman, for whose explanation we are indebted to M. Heins [**16**]. By Theorem III of [**10**], the theorem has applications for certain kinds of integral transforms.

THEOREM II. *Let $u(x)$ and $v(x)$ be square integrable functions defined in* $[0, 1]$, *such that*

$$\bar{u}(x)v(x) = \bar{v}(x)u(x)$$

a.e., and which are essentially linearly independent when restricted to