RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a rcsearch announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON HILBERT'S INEQUALITY IN n DIMENSIONS

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The celebrated inequality of Hilbert asserts that if $a_1, \cdots, a_n$ are real and not all zero then

$$\sum_{\mu,\nu=1}^{n} \frac{a_\mu a_\nu}{\mu+\nu} < \pi \sum_{\nu=1}^{n} a_\nu^2. \tag{1}$$

Furthermore, if the constant π were replaced by a smaller number, then the inequality would be violated for some $n, a_1, \cdots, a_n$. On the other hand, let us regard n as fixed, and let λ_n denote the best possible constant for the inequality

$$\sum_{\mu,\nu=1}^{n} \frac{a_\mu a_\nu}{\mu+\nu} \leqq \lambda_n \sum_{\nu=1}^{n} a_\nu^2. \tag{2}$$

Naturally λ_n is just the largest eigenvalue of the section

$$\left.\frac{1}{\mu+\nu}\right]_{\mu,\nu=1}^{n}$$

of Hilbert's matrix, and we must have

$$0 < \pi - \lambda_n = o(1) \qquad (n \to \infty). \tag{3}$$

The question of obtaining more precise information about the term $o(1)$ in (3) seems to have been first publicly raised by W. W. Sawyer [**1**]. Because of the interest of this problem in numerical analysis [**2**] it has been investigated by several workers [**3**; **4**; **5**] with the result that various upper and lower bounds for the rate of growth of this term are known. We have been able to determine the desired rate of growth more satisfactorily by finding the first two terms of the asymptotic expansion of λ_n. Our result is the following

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