# ON SPECTRAL ESTIMATION

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Inequalities involving linear transformations on Hilbert space and their spectral families will be derived for the purpose of obtaining information about the location of the spectrum. The theorems obtained are extensions of those given by the author [6] and H. D. Block and W. H. J. Fuchs [1] to unbounded transformations for which a spectral decomposition theorem is valid. Such transformations include maximal normal and self-adjoint as well as bounded symmetric and unitary transformations. The domain $\mathfrak{D}_T$ of every linear transformation $T$ being considered is supposed to be dense in Hilbert space $\mathfrak{H}$.

THEOREM 1. *Let $T$ be a self-adjoint linear transformation and let $E(\lambda)$ be the associated spectral family of projections. For an arbitrary real number $\alpha$ and an arbitrary positive number $\epsilon$, let $\Lambda$ denote the open interval $(\alpha-\epsilon, \alpha+\epsilon)$ and let $P(\Lambda)$ denote the projection $E(\alpha+\epsilon-0)-E(\alpha-\epsilon)$. Then for every $x\in\mathfrak{D}_T$ and $v\in E(\alpha-0)\mathfrak{D}_T$, the following inequalities are valid*:

$$\|Tx - \alpha x\| \geqq \epsilon\|x - P(\Lambda)x\|; \tag{1}$$

$$(\alpha v - Tv, v) \geqq \epsilon\|v - P(\Lambda)v\|^2. \tag{2}$$

PROOF. According to the spectral decomposition theorem for self-adjoint transformations [5, p. 180],

$$\|Tx - \alpha x\|^2 = \int_{-\infty}^{\infty} |\lambda - \alpha|^2 d\|E(\lambda)x\|^2 \geqq \epsilon^2 \int_{|\lambda-\alpha|\geqq\epsilon} d\|E(\lambda)x\|^2$$
$$= \epsilon^2\|x - P(\Lambda)x\|^2.$$

This proves (1). To prove (2), let $v=E(\alpha-0)x$ where $x\in\mathfrak{D}_T$. Then $E(\alpha-\epsilon)v=E(\alpha+\epsilon-0)v-P(\Lambda)v=v-P(\Lambda)v$, and the spectral decomposition theorem gives

$$(\alpha v - Tv, v) = \int_{-\infty}^{\alpha} (\alpha - \lambda)d(E(\lambda)v, v) \geqq \epsilon \int_{-\infty}^{\alpha-\epsilon} d(E(\lambda)v, v)$$
$$= \epsilon(E(\alpha - \epsilon)v, v) = \epsilon(v - P(\Lambda)v, v).$$

The first inequality of Theorem 1 generalizes without difficulty to normal transformations [4, p. 355; 5, pp. 311–331]. The proof follows Theorem 1 and will be omitted.