ON A NEW FUNCTIONAL TRANSFORM IN ANALYSIS: THE MAXIMUM TRANSFORM

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1. Introduction. In the study of mathematical economics and operations research, we encounter the problem of determining the maximum of the function

$$F(x_1, x_2, \cdots, x_N) = f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N) \tag{1}$$

over the region R defined by $x_1+x_2+\cdots+x_N=x$, $x_i \geqq 0$. Under various assumptions concerning the f_i, this problem can be studied analytically; cf. Karush [**1**; **2**], and it can also be treated analytically by means of the theory of dynamic programming [**3**].

It is natural in this connection to introduce a "convolution" of two functions f and g, $h=f * g$, defined by

$$h(x) = \max_{0 \leqq y \leqq x} [f(y) + g(x-y)]. \tag{2}$$

For purposes of general study, it is more convenient to introduce instead the convolution $h = f \otimes g$ defined by

$$h(x) = \max_{0 \leqq y \leqq x} [f(y)g(x-y)]. \tag{3}$$

It is easy to see that the operation $\otimes$ is commutative and associative provided that all functions involved are nonnegative. By analogy with the relation between the Laplace transform and the usual convolution, $\int_0^x f(y)g(x-y)dy$, it is natural to seek a functional transform

$$M(f) = F \tag{4}$$

with the property that

$$M(f \otimes g) = M(f)M(g), \tag{5}$$

that is,

$$H(z) = F(z)G(z) \tag{6}$$

where H, F, G are the transforms of h, f, g respectively.

We shall show that M exists and has a very simple form. In addition, M^{-1} has a very simple and elegant representation in a number of cases. More detailed discussions and extensions will be presented subsequently.