MODERN DEVELOPMENTS IN THE GEOMETRY OF NUMBERS[1]

HANS ZASSENHAUS

Modern geometry of numbers is chiefly concerned with two problems, the one is a *packing problem*, the other a *covering problem*. In both cases, an estimate of the density of a subset of some space is required.

The density notion of which I am speaking has its roots in the atomic theory of matter. It may be axiomatized as follows. Let ∂ be a non-negative real valued set function (possibly ∞) on a space E satisfying the conditions

$$\partial(\emptyset) = 0 \qquad (\emptyset \text{ the empty subset}), \tag{1}$$

$$\partial(A \cup B) \geqq \partial(A) + \partial(B) \qquad \text{if } A \cap B = \emptyset. \tag{2}$$

Furthermore assume that there is a permutation group G (symmetry group) of E given. We say that the subset A of E has density $\partial(A)$ if

$$\partial(\alpha A) = \partial(A) \qquad (\alpha \in G), \tag{3}$$

$$\partial(\alpha A \cup B) = \partial(\alpha A) + \partial(B) \qquad \text{if } \alpha A \cap B = \emptyset;\ (\alpha \in G). \tag{4}$$

Thus any symmetry applied to a point set with a density yields a point set of the same density. The union of two disjoint sets both of which have a density has itself a density and the density of the union is equal to the sum of the densities of the summands. For example, let $E = E_n$ be the n-dimensional Euclidean space. Let G be the group of the isometries (rigid movements) of the E_n. For any point set S, denote by $n(t, S)$ the minimum number of points of S belonging to a hypercube:

$$a_i \leqq x_i < a_i + t \qquad (i = 1, 2, \cdots, n).$$

Then

$$\partial(S) = \lim_{t \to \infty} \frac{n(t, S)}{t^n}$$

exists, possibly with $\partial(S) = \infty$ and the properties (1), (2) can be verified.

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[1] An address delivered before the Stanford meeting of the Society on April 22, 1961, by invitation of the Committee to Select Hour Speakers at Far Western Sectional Meetings; received by the editors May 3, 1961.