AN ENCLOSURE THEOREM FOR EIGENVALUES

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THEOREM 1. *Let H be a hermitian matrix and x an arbitrary vector of unit length. Let $\mu=(Hx, x)$, $\sigma=(\|Hx\|^2-\mu^2)^{1/2}$. Then there is an eigenvalue of H in the interval*:

$$\mu - \sigma \leqq \lambda \leqq \mu + \sigma.$$

REMARK. The quantity under the radical is non-negative, since

$$\mu^2 = |(Hx, x)|^2 \leqq \|Hx\|^2.$$

Theorem 1 is a special case of the following theorem.

THEOREM 2. *Let H be a matrix having a complete orthonormal set of eigenvectors. Let x be a vector of unit length. Let $\mu=(Hx,x)$, $\sigma=(\|Hx\|^2-|\mu|^2)^{1/2}$. Then there is an eigenvalue of H in the circle*: $|\lambda-\mu|\leqq\sigma$.

PROOF. Let $x=\sum\xi_i e_i$, where $He_i=\lambda_i e_i$ and $(e_i, e_j)=\delta_{ij}$. Thus $(x, x)=\sum|\xi_i|^2=1$, and $\sigma^2=((H-\mu I)x, (H-\mu I)x)=\sum|\lambda_i-\mu|^2|\xi_i|^2 \geqq |\lambda_m-\mu|^2$, where $|\lambda_m-\mu|=\min_i|\lambda_i-\mu|$. Q.E.D.

Theorem 1 furnishes a simple device for obtaining an interval containing an eigenvalue. As x approaches an eigenvector the interval length (2σ) approaches zero.

This method may be compared with Vazsonyi's enclosure method[1] in which, for a symmetric matrix H, an eigenvalue is guaranteed to lie in the interval

$$\min_i \mu_i \leqq \lambda \leqq \max_i \mu_i,$$

where μ_i is the ratio of the ith component of Hx to the ith component of x. It is an easy exercise to show that $2\sigma\leqq[\max_i\mu_i-\min_i\mu_i]$. In general our interval is considerably smaller than the one obtained by the Vazsonyi method.

The method of Kohn and Kato guarantees, for a symmetric matrix, that an eigenvalue λ_p lies in the interval

$$\mu - \frac{\sigma^2}{\lambda_{p+1}-\mu} \leqq \lambda \leqq \mu + \frac{\sigma^2}{\mu-\lambda_{p-1}},$$

[1] S. H. Crandall, *Engineering analysis*, New York, McGraw-Hill, 1956.