LOCAL CONNECTIVITY IN HOMEOMORPHISM GROUPS

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Recently there has been increasing interest in the local connectivity of the group of all homeomorphisms of a manifold with boundary. Available tools of proof, however, seem to favor the case of a compact manifold [1; 2] or else the use of the topology of uniform convergence of the group [3]. The present note extends such results to the groups of homeomorphisms of certain noncompact manifolds, furnished with the compact-open topology (also see [4]).

If X is a manifold with boundary, let $G(X)$ be the group of homeomorphisms of X, with compact-open topology. $G(X)$ is then a topological transformation group on X. Let the statement that a space X is (respectively) locally connected, locally contractible or locally n-connected be abbreviated by the phase "X is P_h", $h=1$, 2 or 3 respectively.

THEOREM. *Let X be a compact, connected, Hausdorff manifold with boundary,* dim $(X)>1$, *and let F be a finite set of nonboundary points of X. If $G(X)$ is P_h, $h=1$, 2 or 3, then $G(X-F)$ is P_h.*

In particular, this combines with the results of Hamstrom and Dyer [1] to show that if dim $(X)=2$ then $G(X-F)$ is locally contractible; and with the results of Hamstrom [2] to show that if dim $(X)=3$ then $G(X-F)$ is locally n-connected, for all n.

The following lemma will be used in the proof:

LEMMA. *Let X be a compact, connected, Hausdorff manifold with boundary,* dim $(X)>1$; *and let F be a finite set of nonboundary points of X. Then $G(X-F)$ is topologically isomorphic to $G(X, F)$ $=\{g\in G(X): g|F\in G(F)\}$.*

The proof of the lemma is an exercise in the compact-open topology; the hypothesis that X is a manifold is used in an application of the Jordan-Brouwer theorem. This proof is too long to be given here; it will appear elsewhere in another connection.

PROOF OF THE THEOREM. Induction will be used on the number of points of F. Let Y be the set of nonboundary points of X, and let $\{x_j\}$ be a sequence of distinct points of Y. Define $F_i=\bigcup_{j=1}^{i}\{x_j\}$ and $G_i=\{g\in G(X): g(x_j)=x_j \text{ if } x_j\in F_i\}$, with the relative topology.

(i) G_i is a principal fiber bundle over $Y-F_i$ with projection $p: G_i\to Y-F_i: g\to g(x_{i+1})$ and fiber G_{i+1}, for $i=0, 1, \cdots$. The proof of this fact uses the bundle structure theorem: G_{i+1} is a closed subgroup of G_i, and G_i will be a bundle over G_i/G_{i+1} if G_{i+1} has a local