A CONTINUOUS FUNCTION WITH TWO CRITICAL POINTS

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A real C^s-function $f: X \to \mathbf{R}$ on an n-dimensional C^s-manifold with $s \geqq 0$, is called *C^s-nondegenerate C^s-ordinary* at a point $p \in X$, in case a system of n C^s-coordinates (C^s-functions) $\phi_1, \cdots, \phi_n$ exists, which defines a C^s-diffeomorphism κ of some neighborhood $V(p)$ of p into $\mathbf{R}^n$, and such that for some constant $\lambda_p > 0$

$$(1) \phi_i(p) = 0, \; i = 1, \cdots, n; \; \phi_n(q) = \lambda_p\{f(q) - f(p)\}$$

for $q \in V(p) \subset X$.

If C^s-coordinates and $\lambda_p > 0$ exist such that

$$\phi_i(p) = 0, \qquad i = 1, \cdots, n;$$

$$(2) \qquad - \sum_1^r \phi_i^2(q) + \sum_{r+1}^n \phi_j^2(q) = \lambda_p\{f(q) - f(p)\}$$

then the function is called *C^s-critical of index r and C^s-nondegenerate at p.*

A function which is C^s-nondegenerate at every point $p \in X$ is called a *C^s-nondegenerate function.*

We will restrict our considerations to the topological case $s = 0$ of continuous functions on topological manifolds and we will omit C^0 from the notation in the sequel. By *function we will mean continuous function*, etc.

A compact manifold without boundary is called a *closed* manifold. A nondegenerate function on a closed manifold has at least one critical point p_1 of index n and one critical point p_0 of index 0, corresponding respectively with the maximum and the minimum of the function. We prove the

THEOREM. *If X is a closed n-dimensional manifold and $f: X \to \mathbf{R}$ a continuous nondegenerate function with exactly two critical points, then X is homeomorphic to the n-sphere S^n.*[2]

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[2] Reeb [2] proved the corresponding theorem for the differentiable case. Morse [1] proved that X is a homotopy-sphere, and he also has a proof of the theorem we present (unpublished as yet).