CONCENTRIC TORI IN THE 3-SPHERE

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Communicated by R. H. Bing, December 18, 1960

It is proved in this paper that, if A, B, and C are tame solid tori in the 3-sphere S^3 with $A\subset \operatorname{Int} B$ and $B\subset \operatorname{Int} C$, then A and C are concentric if and only if B is concentric with both A and C. It follows from this result that S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric.

A *torus* is the topological product of two circles, while a *solid torus* is the topological product of a circle and a disk. Two solid tori B and B^*, with $B\subset \operatorname{Int} B^*$, are said to be *concentric* if $\operatorname{Cl}(B^*-B)$ is the topological product of a torus and an interval, while two tori T and T^* in S^3 are *concentric* if they are the boundaries of two concentric solid tori B and B^* respectively in S^3. By a *meridianal disk* of the polyhedral solid torus B is meant a polyhedral disk D, with $\operatorname{Int} D \subset \operatorname{Int} B$ and $\operatorname{Bd} D\subset \operatorname{Bd} B$, such that $\operatorname{Bd} D$ is non-nullhomologous on $\operatorname{Bd} B$. Now let D and E be disjoint meridianal disks of the polyhedral solid torus B, and let K_1 and K_2 be the closures of the two components of $B-(D\cup E)$. Suppose that u_1 and u_2 are unknotted polygonal chords of the 3-cells K_1 and K_2 respectively, each with endpoints $x\in \operatorname{Int} D$ and $y\in \operatorname{Int} E$. Then the simple closed polygon $u_1\cup u_2$ is called a *center line* of B.

If k is a simple closed polygon interior to the polyhedral solid torus B in S^3, then the *order of B with respect to k*, denoted by $O(B, k)$, is defined to be the minimal number of points of $k\cap D$, for all meridianal disks D of B [**5**]. If B and B^* are two polyhedral solid tori in S^3, with $B\subset \operatorname{Int} B^*$, then the *order of B^* with respect to B*, denoted by $O(B^*, B)$, is defined to be the order of B^* with respect to an arbitrary center line of B [**5**]. The two polyhedral solid tori B and B^* in S^3 are said to be *equivalently knotted* if and only if any two center lines c of B and c^* of B^* can be so oriented as to represent the same knot (the same equivalence class of oriented closed polygons under orientation-preserving semilinear autohomeomorphisms of S^3) [**5**].

A characterization of the relation of concentricity is provided by

LEMMA 1. *Suppose that B and B^* are two polyhedral solid tori in S^3 with $B\subset \operatorname{Int} B^*$. Then B and B^* are concentric if and only if they are equivalently knotted with $O(B^*, B)=1$.*

Lemma 1 is proved by using some results of Schubert [**5**] on polyhedral solid tori to sharpen the concentric toral theorem of Harrold, Griffith, and Posey [**4**].