SIMULTANEOUS RATIONAL APPROXIMATIONS TO ALGEBRAIC NUMBERS

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Let K be an algebraic number field of degree $n+1$ over the rationals. The conjugates $K^{(0)}, K^{(1)}, \cdots, K^{(n)}$ are arranged so that $K^{(0)}, K^{(1)}, \cdots, K^{(r)}$ are real and

$$K^{(r+s+k)} = \overline{K^{(r+k)}}, \qquad (k = 1, 2, \cdots, s).$$

Here $r+2s=n$. It will be assumed throughout that $r \geqq 0$, so that $K^{(0)}$ is real. Numbers in K are denoted by Greek letters, superscripts being used for the corresponding conjugates. We shall frequently omit the superscript $^{(0)}$; this identification of K with $K^{(0)}$ will cause no confusion. Trace and norm of elements of K are denoted by S and N, respectively.

Let $\beta_0, \cdots, \beta_n$ be elements of K which are linearly independent over the rationals. It is well known that infinitely many sets of rational integers $(q_0, q_1, \cdots, q_n)$ can be found satisfying,

$$(1) \qquad q_0 > 0, \qquad \text{g.c.d.}(q_0, q_1, \cdots, q_n) = 1,$$

and (omitting the superscript $^{(0)}$)

$$(2) \qquad \left| \frac{\beta_j}{\beta_0} - \frac{q_j}{q_0} \right| < C q_0^{-1-1/n}, \qquad (j = 1, \cdots, n),$$

with the constant $C=1$. It will be shown here how to determine all solutions of (1), (2). From this will be deduced not only the known fact that if C is too small (2) has no solutions, but also the hitherto unknown result that the sharper inequalities

$$(3) \qquad \begin{aligned} &| q_0\beta_j - q_j\beta_0 | < C q_0^{-1/n}(\log q_0)^{-1/(n-1)}, \qquad (j = 1, \cdots, n-1), \\ &| q_0\beta_n - q_n\beta_0 | < C q_0^{-1/n}, \end{aligned}$$

have infinitely many solutions.

This result sharpens some of the conclusions of Cassels and Swinnerton-Dyer (I), but does not furnish any further evidence for or against the conjecture of Littlewood which is considered in their paper.

A number of interesting problems can be raised in connection with (3). In one direction it can be asked whether $n-1$ of the inequalities (2) can be improved with factors which are not all the same; e.g.,