ON THE EXTREME EIGENVALUES OF TRUNCATED TOEPLITZ MATRICES

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Let $f(\theta)$ be a real-valued Lesbesgue integrable function defined on $[-\pi, \pi]$. Let $\{C_j\}$ be the Fourier coefficients of $f(\theta)$, i.e.,

$$f(\theta) \sim \sum_{-\infty}^{\infty} C_j e^{ij\theta}.$$

The matrix $T_n[f] = (C_{s-j})$; $s, j = 0, 1, \cdots, n$ is the nth finite section of the infinite Toeplitz matrix (C_{s-j}) associated with the function $f(\theta)$.

In this note we are concerned with functions $f(\theta)$ satisfying

CONDITION A. Let $f(\theta)$ be real, continuous and periodic with period 2π. Let $\min f(\theta) = f(0) = m$ and let $\theta = 0$ be the only value of $\theta (\operatorname{mod} 2\pi)$ for which this minimum is attained.

CONDITION A(α). Let $f(\theta)$ be a function satisfying condition A. Moreover, let $f(\theta)$ have continuous derivatives of order 2α in some neighborhood of $\theta = 0$. Finally let $f^{(2\alpha)}(0) = \sigma^2 > 0$ be the first non-vanishing derivative of $f(\theta)$ at $\theta = 0$.

THEOREM. *Let $f(\theta)$ satisfy conditions A and $A(\alpha)$. Let $\lambda_{\nu,n}$ $(\nu = 1, 2, \cdots, n+1)$ be the eigenvalues of $T_n[f]$ arranged in non-decreasing order. For fixed ν, as $n \to \infty$ we have*

$$\lambda_{\nu,n} = m + \frac{\sigma^2}{(2\alpha)!} \Lambda_\nu \left(\frac{1}{n}\right)^{2\alpha} + o\left(\frac{1}{n}\right)^{2\alpha}, \tag{1}$$

where the numbers Λ_ν are the eigenvalues arranged in nondecreasing order of

$$\left[-\left(\frac{d}{dx}\right)^2\right]^\alpha U - \Lambda U = 0, \qquad 0 \leqq x \leqq 1, \tag{2}$$

with boundary conditions

$$\left(\frac{d}{dx}\right)^i U(0) = \left(\frac{d}{dx}\right)^i U(1) = 0, \qquad i = 0, 1, \cdots, \alpha - 1. \tag{2a}$$

The case $\alpha = 1$ was studied by Kac, Murdock and Szegö [3]. In [5] Widom also studied the case $\alpha = 1$ and, under suitable conditions, obtained the next term in the asymptotic expansion of $\lambda_{\nu,n}$. The case $\alpha = 2$ was studied by this author [4].