ON THE ERGODIC THEOREM WITHOUT ASSUMPTION OF POSITIVITY

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0. **Introduction.** Dunford and Schwartz [**3**; **5**] have recently obtained a generalization of an ergodic theorem of E. Hopf [**4**] to include nonpositive operators in spaces of complex valued functions. Their method of proof is to majorize the operator in question by a positive one, and then to apply Hopf's results. We give a direct proof of a maximal ergodic lemma for operators which are not necessarily positive, in spaces of functions which take their values in an arbitrary Banach space. This lemma is used to further generalize Hopf's theorem so that it applies to operators which are not necessarily positive, in spaces of functions which take their values in a reflexive Banach space. This application of the lemma uses the Kakutani mean ergodic theorem. The method of proof we have used appears to be shorter and simpler than the methods given previously for more special results. Further, our result has the additional advantage that it is sufficiently general to give as a direct corollary a theorem of Kakutani and of Beck and Schwartz [**1**] (stated as Theorem 2 in this paper).

1. **Main results.** Let $\mathfrak{X}$ be a Banach space and (S, Σ, μ) a σ-finite measure space. Let $L_p(S, \Sigma, \mu, \mathfrak{X})$, $1 \leqq p < +\infty$, herein called simply $L_p(S, \mathfrak{X})$ denote the space of all strongly measurable $\mathfrak{X}$-valued functions f defined on S for which the norm given by

$$\|f\|_p = \left(\int_S \|f(s)\|^p(ds)\right)^{1/p} < +\infty;$$

and let $L_\infty(S, \Sigma, \mu, \mathfrak{X})$, herein called simply $L_\infty(S, \mathfrak{X})$, denote the space of all strongly measurable $\mathfrak{X}$-valued functions f defined on S for which the norm g by $\|f(s)\|_\infty = \text{ess sup}_{s\in S} \|f(s)\| < +\infty$.

Let T be a linear operator in $L_1(S, \mathfrak{X})$ such that $\|T\|_1 \leqq 1$, where $\|T\|_1 = \sup_{\|f\|_1 \leqq 1} \|Tf\|_1$, and such that $\|Tf\|_\infty \leqq \|f\|_\infty$ for any function f in $L_1 \cap L_\infty$.

Extend T, if necessary, so that it is defined on each L_p, $1 \leqq p < +\infty$. When convenient we shall suppress the argument of a function, writing f for $f(s)$. Whenever a division by zero is indicated the term in which this appears will be taken to be zero.

If f is a strongly measurable function on S and $a > 0$, let