PROOF OF BLASCHKE'S SPHERE CONJECTURE

BY LEON W. GREEN[1]

Communicated by Hans Samelson, October 17, 1960

Let M be a C^2, compact, orientable two-dimensional Riemannian manifold with positive Gaussian curvature $R(x)$, $x \in M$. Assume that the distance between a point x of M and the first point conjugate to x on any geodesic ray emanating from x is independent of the initial direction and of x. We shall prove the following conjecture of Blaschke:

THEOREM. *M has constant curvature.*

For a brief history of the problem and a proof under stringent additional conditions see [2].

Let us first recall some of the known facts about such surfaces which we shall use. Proofs may be found in [1, §102].

Normalize the conjugate distance so it equals π. Then every geodesic is closed and has length 2π. The mapping of any point into its conjugate point (determined uniquely, independently of the geodesic ray used) is an isometric involution of M onto itself. Any two geodesics intersect in a pair of mutually conjugate points.

Denote the unit tangent bundle of M by T, and set $dK = dA\,d\phi$, where dA is the element of area on M and $d\phi$ is the differential of angle between unit vectors based at the same point. An element of T will be denoted by e, or by the pair (x, ϕ), where ϕ is a fiber coordinate in some local product representation of T; set $p(e) = x$. The geodesic flow in T takes e after time t into the element e_t, the end point of the lift into the bundle of the geodesic segment of length t whose initial element is e. It is well known that dK, the kinematic density, is invariant under this flow.

LEMMA 1.

$$\int_T dK = 8\pi^2.$$

PROOF. Let a closed geodesic G_0 divide M into two domains and call the closure of one of them K_0. Let G_ϵ be the curve swept out by normals to G_0, pointing a distance ϵ into K_0. For sufficiently small

[1] This work was supported by a grant of the National Science Foundation. The author wishes to thank J. Adem for first calling this problem to his attention and C. M. Petty for sending him the report [4].