SOME HILBERT SPACES OF ENTIRE FUNCTIONS

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If $E(z)$ is an entire function such that

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y>0$ $(z=x+iy)$, we write

$$E(z) = A(z) - iB(z)$$

where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})].$$

Let $\mathcal{H}(E)$ be the Hilbert space of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)|^2 |E(t)|^{-2} dt < \infty$$

with integration on the real axis, and

$$|F(z)|^2 \leqq \|F\|^2 K(z, z)$$

for all complex z. This space was introduced in [7] and characterized axiomatically. For each complex number w, $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z and

$$F(w) = \langle F(t), K(w, t)\rangle$$

for each $F(z)$ in $\mathcal{H}(E)$.

The typical example of such a space occurs in Fourier analysis, and then there is a family $(E(a, z))$ of entire functions satisfying (1), $a>0$:

$$E(a, z) = \exp(-iaz),$$

$$A(a, z) = \cos(az), \qquad B(a, z) = \sin(az).$$

When $a \leqq b$, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and every one of these spaces is contained isometrically in $L^2(-\infty, +\infty)$. A necessary and sufficient condition that an entire function $F(z)$ belong to $\mathcal{H}(E(a))$ is that it be of the form

$$\pi F(z) = \int f(t) \cos(tz) dt + \int g(t) \sin(tz) dt,$$

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