SPECTRAL OPERATORS ON LOCALLY CONVEX SPACES

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1. Let C be the complex plane, $S(C)$ the tribe of all Borel parts of C, $B^\infty(C)$ the algebra of bounded complex-valued Borel measurable functions defined on C and $M^1(C)$ the set of bounded complex Radon measures on C. Let E be a locally convex space[1] which is separated, quasi-complete and barrelled. A family $\mathfrak{F}=(m_{x,x'})_{x\in E,x'\in E'}$ of measures belonging to $M^1(C)$ is called a *spectral family* on C if there exists a representation $f\rightarrow U_{\mathfrak{F},f}$ of the algebra $B^\infty(C)$ into the algebra[1] $L(E, E)$ mapping 1 onto I and satisfying the equations $\int_C f dm_{x,x'} = \langle U_{\mathfrak{F},f}x, x'\rangle$ for all $f\in B^\infty(C)$, $x\in E$, $x'\in E'$. By $P_{\mathfrak{F}}$ we denote the *spectral measure* defined on $S(C)$ by the equations $P_{\mathfrak{F}}(\sigma)=U_{\mathfrak{F},\phi_\sigma}$ (ϕ_σ is the characteristic function of σ). A linear mapping T of (the vector space) $D_T\subset E$ into E *commutes* with $\mathfrak{F}$ if $TU_{\mathfrak{F},f}\supset U_{\mathfrak{F},f}T$ for all $f\in B^\infty(C)$.

Let T be a linear mapping of $D_T\subset E$ into E. We say that $\lambda\in\hat{C}$ (=the one point compactification of C) belongs to the *resolvent* set $r(T)$ of T if there is a neighborhood V of λ such that: (i) $zI-T$ is a one-to-one mapping of D_T onto E and $(zI-T)^{-1}\in L(E, E)$ for each $z\in V-\{\infty\}$; (ii) $\{(zI-T)^{-1}\mid z\in V-\{\infty\}\}$ is a bounded part of $L(E, E)$. The set $\mathrm{sp}(T)=\hat{C}-r(T)$ is the *spectrum* of T. If $\mathrm{sp}(T)\not\ni\infty$ we say that T is *regular.*

By an *admissible* set we mean a directed (for $\subset$) set of closed parts of C whose union is C, having a countable cofinal part and containing with $A\subset C$ every closed part of A. We denote below by $\mathfrak{C}_0$ and $\mathfrak{C}_1$ the admissible set of all compact parts of C and all closed parts of C, respectively. Let $\mathfrak{C}$ be an admissible set and T a closed linear mapping of $D_T\subset E$ into E. We say that T is a $\mathfrak{C}$-*spectral operator* if there is a spectral family $\mathfrak{F}$ on C such that:

($\mathrm{D_I}$) T commutes with $\mathfrak{F}$;

($\mathrm{D_{II}}$) $TU_{\mathfrak{F},f}\in L(E, E)$ for each $f\in B^\infty(C)$ whose support is compact and belongs to $\mathfrak{C}$;

($\mathrm{D_{III}}$) $\mathrm{sp}(T_\sigma)\subset\sigma^-$ for every[2] $\sigma\in\mathfrak{C}$.

[1] E barrelled means that every weakly bounded part of the dual space E' is equicontinuous; E quasi-complete means that every bounded closed part of E is complete. $L(E, E)$ is the algebra of all linear continuous mappings of E into E endowed with the topology of uniform convergence on the bounded parts of E.

[2] For a set $A\subset C$ we denote by A^- the closure of A in $\hat{C}$.