KNOTTING MANIFOLDS

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We give a definition of isotopy and of knotting of an arbitrary space in an arbitrary space. The definition is of most interest when the spaces are manifolds. To avoid pathology, and to obtain theorems, we confine ourselves to combinatorial theory.

DEFINITIONS. Let Y be a finite simplicial complex, and let I denote the unit interval. An *isotopy* of Y is a piecewise linear homeomorphism $h: I\times Y \to I\times Y$ such that

(i) $h(t\times Y) = t\times Y$, $0 \leqq t \leqq 1$; (therefore for each t there is a piecewise linear homeomorphism $h_t: Y\to Y$, such that $h(t, y) = (t, h_t y)$);

(ii) h_0 is the identity on Y.

If X is another complex, two piecewise linear embeddings $f, g: X\to Y$ are *isotopic* if there is an isotopy h of Y such that $h_1 f = g$. Isotopy is an equivalence relation on the set of all piecewise linear embeddings of X in Y. Let $\mathrm{Iso}(X\subset Y)$ denote the set of equivalence classes. There is a natural map from $\mathrm{Iso}(X\subset Y)$ onto $\mathrm{Hom}(X\subset Y)$, the set of homotopy classes of piecewise linear embeddings, because isotopic maps are clearly homotopic. If this map is one-to-one we say that X *unknots* in Y; otherwise we say X *knots* in Y.

EXAMPLES. (i) The circle knots in the 3-sphere, as is well known, because there are many isotopy classes but only one homotopy class.

(ii) A point unknots in a closed manifold, and knots in a bounded manifold, because there is one isotopy class for the interior of the manifold, and one for each boundary component. For the rest of this paper, however, we shall confine ourselves to closed manifolds. (There is a relative theory which is more appropriate for bounded manifolds.)

MOTIVATION. We choose this definition of isotopy, because it seems the most natural intuitive concept associated with the notion of piecewise linear embedding. In effect, we say that two embeddings f, g of X in Y are equivalent if we can, not only slide fX onto gX, but slide the pair $fX\subset Y$ onto the pair $gX\subset Y$, through a continuous family of equivalent piecewise linear embeddings. Our definition is stronger than Mazur's ambient isotopy, since it involves the embeddings f, g specifically (see [6, §§2, 3]). It is in fact the strongest available combinatorial definition of isotopy, and so Theorems 1 and 2 below are in their most powerful form.