CURVATURE AND LENGTH

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In this note an invariance property of curvature of plane curves is discussed. The property is essentially this: the pointwise curvature of a family F of plane curves integrated over the interior of a simple, closed, smooth curve B depends only upon the angles of intersection of B with members of F. The property is exactly stated and proved below, and some of its consequences are discussed. These are first, the connection of the integrated curvature with the boundary length, and second an integral inequality for meromorphic functions.

The results depend upon the following simple lemma.

LEMMA A. *If u is a real-valued function of class C^2 defined in the plane, $K(x, y)$ is the curvature at (x, y) of the level curve $u=$*const. *through (x, y), and $\mathbf{n}$ is the unit normal to $u=$*const. *then*

$$\operatorname{div} \boldsymbol{n} = K.$$

The proof follows readily by calculation of the divergence of n from

$$\boldsymbol{n} = (u_x, u_y)/(u_x^2 + u_y^2)^{1/2},$$

and comparison with the easily obtained formula.

$$K = (u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx})/(u_x^2 + u_y^2)^{3/2}.$$

We take the sign of K to be defined by this formula.

With this lemma we are able to prove the following theorem on the integrated curvature over a region for such a function.

THEOREM 1. *If G is a region in the plane whose boundary B consists of a finite number of simple, closed, piecewise smooth curves, and u is a function of class C^2 almost everywhere whose level curves have curvature $K(x, y)$, then*

$$\int_G K dA = \int_B \boldsymbol{n} \cdot \boldsymbol{N} ds,$$

*where $\mathbf{n}$ and $\mathbf{N}$ are the unit normal vectors to $u=$*const. *and B.*

The proof is an application of Green's Theorem. If, using Lemma A, we replace K by div $\boldsymbol{n}$ then the above formula is exactly a statement of Green's Theorem. Of course, the conditions placed on the