ON SOME FUNCTIONS OF LITTLEWOOD-PALEY AND ZYGMUND

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Communicated by P. R. Halmos, August 21, 1960

In a previous paper [2], we studied the n-dimensional form of the functions of Littlewood-Paley and Lusin. These were defined as follows. Let $f(x)\in L^p(E_n)$, E_n is Euclidean n-space, of variables $x, y, \cdots, x=(x_1, x_2, \cdots, x_n)$; let $U(x, t)$, $t>0$, be the Poisson integral of f. Let

$$g(x) = \left(\int_0^\infty t\,|\nabla U|^2 dt\right)^{1/2} \text{ and } S(x) = \left(\iint_{W(x)} t^{1-n}\,|\nabla U|^2 dt dy\right)^{1/2}.$$

Here

$$|\nabla U|^2 = \sum_{k=0}^{n} \left(\frac{\partial U}{\partial x_k}\right)^2, \qquad x_0 = t;$$

$W(x)$ is the cone $\{(y, t): |x-y| < \alpha t\}$. We proved

$$(1) \qquad B_p\|f\|_p \leqq \|g\| \leqq A_p\|f\|_p, \qquad 1 < p < \infty,$$

with a similar result for S.

We wish now to consider a related function of Littlewood-Paley and Zygmund. We define its n-dimensional version as follows. Let $0<\lambda$, and set

$$g_\lambda^*(x;f) = g_\lambda^*(x) = \left(\int_0^\infty \int_{E_n} \frac{t^{\lambda+1}}{(|x-y|^2+t^2)^{(\lambda+n)/2}}\,|\nabla U|^2 dy dt\right)^{1/2}.$$

We note first

$$g(x) \leqq AS(x) \leqq B_\lambda g_\lambda^*(x).$$

The first part of this inequality is Lemma 9 of [2], and the second part is trivial. We note also that $g^*_{\lambda_1}(x) \leqq g^*_{\lambda_2}(x)$, if $\lambda_2 \leqq \lambda_1$. We shall see that the behavior of g_λ^* when $\lambda > n$ is similar to that of the simpler functions g and S. Hence our primary concern will be with g_λ^* when $0<\lambda\leqq n$. We outline the proof of the following theorem.

THEOREM. *Let $0<\lambda\leqq n$, and $2n/(\lambda+n)<p<\infty$. Then*

$$\|g_\lambda^*\|_p \leqq A_{p,\lambda}\|f\|_p, \qquad A_{p,\lambda} \textit{ independent of } f.$$