THE GENERALIZED POINCARÉ CONJECTURE IN HIGHER DIMENSIONS

BY STEPHEN SMALE[1]

Communicated by Edwin Moise, May 20, 1960

The Poincaré conjecture says that every simply connected closed 3-manifold is homeomorphic to the 3-sphere S^3. This has never been proved or disproved. The problem of showing whether every closed simply connected n-manifold which has the homology groups of S^n, or equivalently is a homotopy sphere, is homeomorphic to S^n, has been called the generalized Poincaré conjecture.

We prove the following theorem.

THEOREM A. *If M^n is a closed differentiable (C^∞) manifold which is a homotopy sphere, and if $n \neq 3, 4$, then M^n is homeomorphic to S^n.*

We would expect that our methods will yield Theorem A for combinatorial manifolds as well, but this has not been done.

The complete proof will be given elsewhere. Here we give an outline of the proof and mention other related and more general results.

The first step in the proof is the construction of a nice cellular type structure on any closed C^∞ manifold M. More precisely, define a real valued f on M to be a *nice* function if it possesses only nondegenerate critical points and for each critical point β, $f(\beta)=\lambda(\beta)$, the index of β.

THEOREM B. *On every closed C^∞ manifold there exist nice functions.*

The proof of Theorem B is begun in our article [3]. In the terminology of [3], it is proved that a gradient system can be C^1 approximated by a system with stable and unstable manifolds having normal intersection with each other. This is the announced Theorem 1.2 of [3]. From this approximation we are then able to construct the function of Theorem B.

The stable manifolds of the critical points of a nice function can be thought of as cells of a complex while the unstable manifolds are the duals. This structure has the advantage over previous structures that both the cells and the duals are differentiably imbedded in M. We believe in fact that nice functions will replace much of the use of C^1 triangulations and combinatorial methods in differential topology.

[1] Supported by a National Science Foundation Postdoctoral fellowship.