HOMOTOPY-ABELIAN LIE GROUPS

BY S. ARAKI, I. M. JAMES AND EMERY THOMAS[1]

Communicated by Hans Samelson, May 12, 1960

A topological group G is said to be *homotopy-abelian* if the commutator map of $G\times G$ into G is nulhomotopic. Examples can be given[2] of non-compact Lie groups which are homotopy-abelian but not abelian. The purpose of this note is to prove

THEOREM. *A compact connected Lie group is homotopy-abelian only if it is abelian.*

COROLLARY. *If a Lie group is homotopy-abelian, then its maximal compact connected subgroup is abelian.*

Our proof depends on the theory of [6]. Thus we consider the Samelson "commutator" product[3] in the homotopy groups of G, which is trivial when G is homotopy-abelian. The product of $\alpha\in\pi_p(G)$ with $\beta\in\pi_q(G)$ is denoted by $\langle\alpha, \beta\rangle\in\pi_{p+q}(G)$, where $p, q\geqq 1$. If h is a homomorphism of G into another topological group then

$$h_*\langle\alpha, \beta\rangle = \langle h_*\alpha, h_*\beta\rangle,$$

where h_* denotes the induced homomorphism. Note that h_* is an isomorphism if h is a covering map and $p, q\geqq 2$. Hence if two topological groups have a common universal covering group then their higher homotopy groups are related by an isomorphism which is compatible with the Samelson product. Let $\sigma\,\pi_q(G)$, where $q\geqq 1$, denote the subset of $\pi_{2q}(G)$ consisting of elements $\langle\beta, \beta\rangle$, where $\beta\in\pi_q(G)$. We assert the following

LEMMA. *Let G be a compact connected simple non-abelian Lie group of dimension n and rank l. Then $\sigma\,\pi_q(G)\neq 0$, where $q=2n/l-3$.*

The proof is by application of (2.2) of [6]. We distinguish between the classical and exceptional cases, beginning with the latter.

Let G be one of the exceptional groups. Then $n/l=p$, an odd prime number, and G has no p-torsion (see [3]). The mod p cohomology of G is an exterior algebra on a basis of l generators. There is one generator y in dimension q, while the remainder are of lower dimension. It follows from Proposition 6 on page 291 of [8] that y has a nontrivial

---

[1] Research supported in part by U.S. Air Force Contract AF 49(638)–79.

[2] Such as the 2-dimensional affine group (example suggested by H. Samelson).

[3] The theory of the Samelson product is given in [5], for example.