AN ARITHMETICAL INVERSION PRINCIPLE

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Let $f(n, r)$ represent an even function of n (mod r); that is, $f(n, r) = f((n, r), r)$ for all integers n and a positive integral variable r. The following inversion relation is proved in [2]. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(1) \qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \mu(d) \rightleftarrows f(n, r) = \sum_{d \mid r} g\left(d, \frac{r}{d}\right),$$

where $\mu(r)$ denotes the Möbius function. This relation can be easily verified on the basis of the definition of even function (mod r) and the characteristic property of $\mu(r)$,

$$(2) \qquad \sum_{d \mid r} \mu(d) = \epsilon(r) \equiv \begin{cases} 1 & (r = 1), \\ 0 & (r > 1). \end{cases}$$

We now state a generalization of (1). Let $\xi(r)$ and $\eta(r)$ be arithmetical functions satisfying

$$(3) \qquad \sum_{d\delta = r} \xi(d)\eta(\delta) = \epsilon(r).$$

The following theorem can be proved in the same manner as (1), with (3) used in place of (2).

THEOREM 1. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(4) \qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \eta(d) \rightleftarrows f(n, r) = \sum_{d\delta = (n, r)} g\left(d, \frac{r}{d}\right) \xi(\delta).$$

Clearly (4) reduces to (1) in case $\xi(r) = 1$, $\eta(r) = \mu(r)$. The case $\xi(r) = \mu(r)$, $\eta(r) = 1$ yields the following dual of (1).

THEOREM 2. *If* $r = r_1 r_2$ *and* $f(n, r)$ *is even* (mod r), *then*

$$(5) \qquad g(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \rightleftarrows f(n, r) = \sum_{d\delta = (n, r)} g\left(d, \frac{r}{d}\right) \mu(\delta).$$

An immediate consequence of Theorem 2 is

COROLLARY 2.1. *For every arithmetical function* $g(r_1, r_2)$ *of two positive integral variables* r_1, r_2, *there exists a uniquely determined even function* (mod r), $f(n, r)$, *such that* $g(r_1, r_2)$ *is expressible as a divisor sum* (5) *with respect to* $f(n, r)$.